Computability and Totality in Domains

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Abstract

We survey the main results on computability and totality in Scott-Ershov-domains as well as their applications to the theory of functionals of higher types and the semantics of functional programming languages. A new density theorem is proved and applied to show the eqivalence of the hereditarily computable total continuous functionals with the hereditarily effective operations over a large class of base types.

1 Introduction

This paper studies different concepts of computability on partial and total continuous functionals of finite types. Three computability concepts will be studied 1. definability by a program in a functional language, 2. effective continuity, and 3. computability via a recursive transformation of codes (effective operations). For 3. to make sense the hereditarily effective versions of the hierarchies have to be considered. Although apparently fundamentally different in nature these concepts turn out to be equivalent on both hierarchies, provided in 1. an appropriate language is chosen. Plotkin [37] has shown the equivalence of 1. and 2. on the partial continuous functionals w.r.t. the language PCF augmented by the parallel conditional and the parallel $\exists$. Normann [34] proved the corresponding equivalence for the total continuous functionals and pure PCF (without parallel facilities). The equivalence of 2. and 3. is due to Ershov [13], for the partial and the total case. For the partial case a similar result has been obtained by Constable and Egli [9].

Following Ershov [13] we define the partial and total continuous functionals in the framework of effective Scott-Ershov domains. Ershov showed that his total continuous functionals are isomorphic to those defined by Kleene [20] and Kreisel [23], and that domain-theoretic computability (i.e. 2.) corresponds to recursive countability (i.e. having a recursive associates). The domain-theoretic approach not only allows for a very elegant definition of the

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continuous functionals, but also gives deeper insights into the phenomenons being studied. In fact most of the results mentioned so far rest on quite general theorems on computability and totality in domains. For instance the density of the total continuous functionals used in Normann's proof is an instance of a general domain-theoretic theorem [3], and Ershov's results are instances of domain-theoretic generalizations of the well-known Myhill-Shepherdson and Kreisel-Lacombe-Shoenfield [13], [3]. The main new result of our paper follows this line. We prove a density theorem for abstract domains with totality generalising the corresponding theorem in [3] and also a recent result of Normann's showing density for the total continuous functionals over the reals [33]. In fact our proof is inspired by Normann's proof. Using the domain theoretic generalisation of the Myhill-Shepherdson and Kreisel-Lacombe-Shoenfield theorem mentioned above we conclude that 2. and 3. are equivalent also for the partial and total continuous functionals over the reals. The equivalence of 1. and 2. for these functionals has been shown by Escardo [15] using a version of PCF, called Real PCF, extended by the parallel $\exists$. The corresponding problem for the total continuous functionals over the reals and Real PCF (without $\exists$) seems to be open.

We will also briefly discuss extensions of the partial and total continuous functionals to dependent and transfinite types [35], [31], [4]. The equivalence of 2. and 3. for these types has been shown by Normann and Waagbø [45] using results from [4].

2 Computability in Scott-Ershov-domains

In this section we prove the equivalence of the concepts 1. 2., and 3. discussed in the introduction for the partial continuous functionals.

In order to fix notation we recall some basic definitions concerning domains, mainly following [18]. By a *Scott-Ershov-domain* we mean a partially orderd set $(D, \sqsubseteq)$ which is *directed complete*, i.e. every directed set $A \subseteq D$ has a least upper bound $\bigsqcup A \in D$ (A is *directed* if $A \neq \emptyset$ and $\forall x, y \in A \exists z \in A(x, y \sqsubseteq z)$), *algebraic*, i.e. for every $x \in D$ the set $\{x_0 \in D: x_0 \text{ compact and } x_0 \sqsubseteq x\}$ is directed and has x as its least upper bound ($x_0 \in D$ is *compact* if for every directed set $A \subseteq D$ such that $x \sqsubseteq \bigsqcup A$ we have $x_0 \sqsubseteq y$ for some $y \in A$), *countably based*, i.e. the set of compact elements is countable, *bounded complete*, i.e. every nonempty bounded subset of D has a least upper bound in D, equipped with a *least element*, usually denoted $\perp$. We will assume in addition that all Scott-domains in consideration are *coherent*, which means that a nonempty subset is bounded whenever all its two element subsets are bounded. Although all results presented in this paper also hold without this assumptions, many notions have an easier definition and some proofs become less clumsy when coherency is assumed. The set of compact elements of a Scott-domain D is denoted by D_0. The *Scott-topology* on D is generated by the basic open sets $\{x \in D: x \sqsupseteq x_0\}$, where $x_0 \in D_0$. Tho elements $x, y \in D$

are called *consistent*, written $x \uparrow y$, if $\{x, y\}$ is bounded in D. Clearly this is the case if and only if x and y cannot be separated by disjoint neighbourhoods.

An *effective Scott-domain* is a Scott-domain $(D, \sqsubseteq)$ together with a numbering $\nu_0 : \mathbf{N} \to D_0$, called *effectivation*, such that

1. the sets $\{(n,m) \mid \nu_0 n \sqsubseteq \nu_0 m\}$, and $\{(n,m) \mid \nu_0 n \sqcup \nu_0 m \text{ exists}\}$ are decidable,
2. there is a recursive function $f : \mathbf{N} \times \mathbf{N} \to \mathbf{N}$ such that $\nu_0 f(n,m) = \nu_0 n \sqcup \nu_0 m$ whenever the supremum exists.

An element $x \in D$ is *computable* iff the set $\{n \mid \nu_0 n \sqsubseteq x\}$ is recursively enumerable. We let D_{comp} denote the set of computable elements of D.

Convenbtion: In the following effective Scott-domains will be simple called *domains*. We let the letters D, E, F range over domains.

Basic examples of domains are the flat domains $\mathbf{N}_\bot := \mathbf{N} \cup \{\bot\}$ and $\mathbf{B}_\bot := \mathbf{B} \cup \{\bot\}$ ($\mathbf{B} = \{\mathtt{tt}, \mathtt{ff}\}$) of *partial integers* and *boolean values*. We will also consider the domain R of *partial reals*. R is the ideal completion of the partial order

$$I_{\mathbf{Q}} := \{[a,b] \mid a \in \{-\infty\} \cup \mathbf{Q},\, b \in \mathbf{Q} \cup \{+\infty\},\, a \le b\},$$

where $\mathbf{Q}$ is the set of rational numbers. The ordering on $I_{\mathbf{Q}}$ corresponds to reverse inclusion of closed intervals, i.e. $[a,b] \sqsubseteq [a',b']$ iff $a \le a'$ and $b' \le b$. The elements of R are downward closed directed subsets $A \subseteq I_{\mathbf{Q}}$ (ideals). The ordering on R is set inclusion. In ideal $A \in R$ which is 'converging', i.e. $\delta(A) := \inf\{b - a \mid [a,b] \in A\} = 0$ in a natural way represents the real number $r := \sup\{a \mid [a,b] \in A\} = \inf\{b \mid [a,b] \in A\}$.

It is well-known that domains and continuous function form a cartesian closed category. A continuous function between domains is called *effectively continuous* is it is a computable element of the function space.

The cartesian closed subcategory generated by the domains $\mathbf{N}_\bot$ and $\mathbf{B}_\bot$ is usually called the hierarchy of *partial continuous functionals of finite types*. The objects of this category are a family of domains D_ρ, where

$$D_\iota := \mathbf{N}_\bot, \quad D_o := \mathbf{B}_\bot, \quad D_{\rho\times\sigma} := D_\rho \times D_\sigma, \quad D_{\rho\to\sigma} := D_\rho \to D_\sigma.$$

Henceforth τ will refer to one of the base types ι and o, and for brevity we will ignore product types.

The notion of an effective operation (i.e. concept 3.) refers to a standard numbering $\nu : \mathbf{N} \to D_{\mathrm{comp}}$ of the computable elements of a domain $(D, \sqsubseteq, \nu_0)$ called *principal constructivation*. It always exists and is characterized up to recursive equivalence by the conditions

1. the set $\{(n,m) \mid \nu_0 n \sqsubseteq \nu m\}$ is recursively enumerable,
2. there is a recursive function $g : \mathbf{N} \to \mathbf{N}$ such that $\nu_0 n = \nu g(n)$ for all n,
3. For any other numbering $\nu' : \mathbf{N} \to D_{\mathrm{comp}}$ satisfying 1. and 2. there is a recursive function $h : \mathbf{N} \to \mathbf{N}$ such that $\nu' n = \nu h(n)$ for all n.

An *effective operation between domains* is a function $F : D_{\mathrm{comp}} \to E_{\mathrm{comp}}$ which is tracked by some recursive function $f: \mathbf{N} \to \mathbf{N}$, i.e. $F(\nu n) = \mu f(n)$ for all $n \in \mathbf{N}$. Note that this definition doesn't require F to be continuous.

It is easy to see that a continuous function between domain is effectively continuous iff its restriction to computable arguments is an effective operation. Therefore, in order to prove the equivalence of the computability concepts 2. and 3. it remains to show that every effective operation is continuous. The latter follows from domain-theoretic generalisations of two theorems from elementary recursion theory. They establish a surprising connection between recursion theory and topology. The generalisation to domain theory is due to Ershov [13].

Theorem 1 (Generalized Rice-Shapiro Theorem) *Let $U \subseteq D_{comp}$ be such that the set $\{n \mid \nu n \in U\}$ is recursively enumerable. Then U is an open subset of D_{comp} (w.r.t. the relativized Scott-topology).*

This theorem can be proved by either employing a recursively enumerable nonrecursive set, or by using the recursion theorem. If one is interested in constructive meta theory it is important to note that the proof requires Markov's principle.

Theorem 2 (Generalized Myhill-Shepherdson Theorem) *Every effective operation between domains is continuous (w.r.t. the relativized Scott-topologies).*

This can be used to show that the hierarchy of effectively partial continuous functionals coincides with the hierarchy of effective operations based on a principal constructivation of $\mathbf{N}^{\perp}$ (e.g. $\lambda n, \{(n)_0\}(n)_1$) [2].

Theorem 3 *The effective partial continuous functionals and the partial effective operations over $\boldsymbol{N}$ are effectively isomorphic.*

Now we turn our attention to the concepts 1. and 2. It is well-known that the partial continuous functionals form a model for the functional programming language PCF [37] and also for Kleene's schemes (S1-S9).

The following theorem is due to Platek [36] (first part) and Plotkin [37] (second part).

Theorem 4 *On the partial continuous functionals PCF-definability and (S1-S9) computability coincide, but are weaker than domain theoretic computability.*

Hence for PCF and (S1-S9) equivalence of 1. and 2. does not hold. The reason for this is that PCF is unable to define such simple functions like the parallel or, POR: $\mathbf{B}_\perp \to \mathbf{B}_\perp \to \mathbf{B}_\perp$ returning $\mathtt{tt}$ as soon as one of its argument is $\mathtt{tt}$ (whereas the other may be undefined, i.e. $\perp$) [37]. This can be remedied by either restricting the continuous functionals to some 'sequential' fragment [26], or extend PCF. Plotkin [37] showed how to do the latter. He proved that besides POR only one further functional $\exists: (\mathbf{N} \to \mathbf{B}_\perp) \to \mathbf{B}_\perp$ is needed, defined by $\exists(f) = \mathrm{tt}$ is $f(n) = \mathrm{tt}$ for some $n \in \mathbf{N}$, and $\exists(f) = \mathrm{ff}$ if $f(\bot) = \mathrm{ff}$.

Theorem 5 *In PCF+POR every compact functional is definable and hence the partial continuous functionals form a fully abstract model for PCF+POR. Adding further the parallel existential quantifier $\exists$ yields full domain theoretic computability.*

In [14] Escardø introduced *Real PCF*, an extension of PCF by a base type for the partial real numbers, and interpreted it in a corresponding extension of the partial continuous functionals by the domain R of partial reals. In [14] he extends Plotkin's result to this situation.

Theorem 6 *In Real PCF+$\exists$ every effectively continuous functional over the partial reals is definable.*

To be precise, Escardø works with continuous domains, but, since continuous domains are retracts of domains, it is fairly obvious how to translate his results to uor framework. The corresponding problem for the total continuous functionals over the reals and Real PCF (without $\exists$) seems to be open.

3 Totality

In [13] the total continuous functionals are defined as follows. For every type ρ define $\overline{D}_\rho \subseteq D_\rho$ by

$$\overline{D}_\iota := \mathbf{N}, \quad \overline{D}_o := \mathbf{B}, \quad \overline{D}_{\rho\to\sigma} := \{f \in D_{\rho\to\sigma} | f[\overline{D}_\rho] \subseteq \overline{D}_\sigma\}$$

and define equivalence relations $=_\rho$ on $\overline{D}_\rho$ by

$$x =_\tau y :\Leftrightarrow x = y, \quad f =_{\rho\to\sigma} g :\Leftrightarrow \forall x \in \overline{D}_\rho f(x) =_\sigma g(x)$$

Then the total continuous functionals are the equivalence classes of the $\overline{D}_\rho$. From the fact that $\overline{D}_\rho$ is dense in D_ρ (see below) it follows that for $x, y \in \overline{D}_\rho$

$$x =_\rho y \Leftrightarrow x \uparrow y$$

Hence $=_\rho$ is respected by application, i.e. application on the quotient structure is well-defined. Instead of the types $\mathbf{N}_\bot$ and $\mathbf{B}_\bot$ with total elements $\mathbf{N}$ and $\mathbf{B}$ also other domains D and selected subsets $\overline{D}$ could be use as base types. The resulting hierarchy $\overline{D}_\rho$ will then be called *total continuous functionals over* $\overline{D}$. Furthermore the function space can be hereditarily restricted to computable elements,

$$\overline{D}^{\mathrm{comp}}_{\rho\to\sigma} := \{f \in D_{\rho\to\sigma} \mid f \text{ computable and } f[\overline{D}^{\mathrm{comp}}_\rho] \subseteq \overline{D}^{\mathrm{comp}}_\sigma\}.$$

We will call this the *hereditarily computable total continuous functionals over* $\overline{D}$

In this sections we establish general conditions on $(D, \overline{D})$ under which the construction of the (herediterily computable) total continuous functionals over $(D, \overline{D})$ is possible. In particular we will be interested in proving the crucial density property in all types.

Abstracting from Ershov's approach to the total continuous functionals Normann proposed the notion of a *domain with totality* [32]. For our purposes we will slightly modify his definitions. A pair $(D, \overline{D})$ where D is a domain and $\overline{D}$ is a subset of D is called a *domain with totality*. The set $\overline{D}$ is called the totality on D and the elements in $\overline{D}$ are called *total*. Sometimes we will refer to $(D, \overline{D})$ simply as $\overline{D}$ as long as this doesn't cause ambiguities. The totality $\overline{D}$ is called *strong* if the consistency relation $\uparrow$ on $\overline{D}$ is an equivalence relation. If $\overline{D}$ is strong then for every total x we let $\mathbf{x} := \{y \in \overline{D} \mid x \uparrow y\}$, the equivalence class of x. Furthermore we let $\mathbf{D} := \overline{D}/\!\uparrow = \{\mathbf{x} \mid x \in \overline{D}\}$ denote the quotient structure endowed with the quotient topology. An equivalence class $\mathbf{x} \in \mathbf{D}$ is called *computable* if it contains a computable element. More general if P is a property of elements of $\overline{D}$ then $\mathbf{x} \in \mathbf{D}$ is said to have property P if some element of $\mathbf{x}$ has property P.

For every domain the set of its maximal elements is a strong totality. Further examples of domains with strong totality are (as we will see) the continuous functionals over $\mathbf{N}$ and $\mathbf{B}$ and also their hereditarily computable versions. As shown by Ershov [12] their quotients define the Kleene-Kreisel (hereditarily computable) total continuous functionals. The computable elements of $\mathbf{D}_\rho$ correspond to the recursively continuous functionals. The domain R of partial reals becomes a domain with strong totality by setting $\overline{R} := \{A \in R \mid \delta(A) = 0\}$. The quotient space $\mathbf{R}$ is homeomorphic to the reals. [6] and [7] contain many further examples of interesting topological spaces represented in the form $\mathbf{D}$.

If $(D, \overline{D})$ and $(E, \overline{E})$ are domains with totality then the total elements of $D \to E$ are defined by

$$\overline{D \to E} := \overline{D} \to \overline{E} := \{f \in D \to E \colon f[\overline{D}] \subseteq \overline{E}\}.$$

The elements of $\overline{D} \to \overline{E}$ are called *total functions*. However in general $\overline{D} \to \overline{E}$ will not be strong even if $\overline{D}$ and $\overline{E}$ both are. Moreover it is natural to consider $f, g \in \overline{D} \to \overline{E}$ as equivalent if $f(x) \uparrow g(x)$ for all $x \in \overline{D}$, but this notion of equivalence will in general not coincide with $\uparrow$ in $\overline{D} \to \overline{E}$.

This can be remedied by requiring the total elements to be dense. Note that $\overline{D} \subseteq D$ is dense iff

$$\forall x_0 \in D_0 \, \exists x \in \overline{D} \; x_0 \sqsubseteq x.$$

One immediately checks that if D, E are domains with totality such that $\overline{D}$ is dense and $\overline{E}$ is strong, then $\overline{D} \to \overline{E}$ is strong. Moreover for $f, g \in \overline{D} \to \overline{E}$ we have $f \uparrow g$ iff $f(x) \uparrow g(x)$ for all $x \in \overline{D}$. The latter amounts to a principle of *extensionality*: Two total functions are identified if they are extensionally equal on total arguments, that is, for $f, g \in \overline{D} \to \overline{E}$ we have $\mathbf{f} = \mathbf{g}$ iff $\mathbf{f}(\mathbf{x}) = \mathbf{g}(\mathbf{x})$ for all $x \in \overline{D}$, where, of course, $\mathbf{f}(\mathbf{x})$ is the equivalence class of $f(x)$.

We are still not satisfied, since, even if $\overline{D}$ and $\overline{E}$ are both strong and dense, $\overline{D} \to \overline{E}$ need not be dense. Consider for example $D := \mathbf{N}^\perp$ with $\overline{D} := D$, and $E := \mathbf{N}^\perp$ with $\overline{E} := \mathbf{N}$. Both are strong and dense totalities, but $\overline{D} \to \overline{E}$ contains only constant functions and hence is not dense. What is wrong here is the fact that we declared $\perp \in D$ to be total. To exclude this we obviously need a property of $\overline{D}$ forcing the elements of $\overline{D}$ to be in some sense 'large'. In [3] the notion of *co-ensity* was introduced which together with density was preserved under function spaces. This solved the density problem for functionals over discrete base types like $\mathbf{N}$ or $\mathbf{B}$, but unfortunately it excluded base types like the reals, since codensity of $\overline{D}$ implies that the quotient space $\mathbf{D}$ is strongly disconnected in the sense that any two different points can be separated by clopen sets.

In the following we introduce generalisations of the notions of density and co-density and prove a density theorem generalising the results in [3] and also Normann's density theorem for the total continuous functionals over the reals [33]. In fact our proof has been obtained by an analysis of Normann's proof.

For any domain D we let D^ω denote the set of functions $s\colon \mathbf{N} \to D$. Ordered pointwise this is again a domain. Any totality $\overline{D}$ on D gives rise to the totality $\overline{D}^\omega := \{s \in D^\omega \mid s[\mathbf{N}] \subseteq \overline{D}\}$ on D^ω. $\overline{D}^\omega$ will be strong if $\overline{D}$ is. Note that because domains have a countable base, the totality $\overline{D}$ is dense in D iff there is an $s \in \overline{D}^\omega$ such that $s[\mathbf{N}]$ is dense in D. We call $\overline{D}$ *effectively dense* if there is a computable such s. Since we will be interested in effective density only we will henceforth by 'dense' always mean 'effectively dense'

A continuous function $f\colon D \to E$ is called *separating* if it preserves inconsistencies, i.e. $\forall x, y \in D\,(x \not\uparrow y \Rightarrow f(x) \not\uparrow f(y))$.

Let $(A, \overline{A})$ be a domain with totality. A totality $\overline{D}$ on a domain D is called $\overline{A}$-dense if $\overline{A} \to \overline{D}$ is dense in $A \to D$. $\overline{D}$ is called *A-codense* if $\overline{D} \to \overline{A}$ contains a computable separating element.

For example if $\overline{A} := \overline{D}^\omega$ then $\overline{A}^\omega$ is $\overline{A}$-codense since A^ω and A are isomorphic via $\mathrm{melt}\colon A^\omega \to A$, $\mathrm{melt}(s)(<m,n>) := s(m)(n)$, where $<.,.>$ is some primitive recursive bijection from $\mathbf{N}^2$ to $\mathbf{N}$. Clearly melt is total and computable. Furthermore it is separating because it is an isomorphism.

The following facts whose simple proofs we omit shed some light on the notions of A-density and A-codensity, and relate them to previous definitions.

1. $\overline{D}$ is dense iff it is $\mathbf{B}^\omega$-dense.
2. $\overline{D}$ is codense (in the sense of [3]) iff it is $\mathbf{B}^\omega$-codense.
3. If $\overline{D}$ is $\overline{A}$-dense and $\overline{A}$ is nonempty then $\overline{D}$ is dense.
4. If $\overline{D}$ is $\overline{A}$-codense and $\overline{A}$ is strong then $\overline{D}$ is strong.

Theorem 7 *Let $(A, \overline{A})$, $(D, \overline{D})$, and $(E, \overline{E})$ be domains with totality. Assume that there is some computable total and separating function melt $\in A^\omega \to A$ (i.e. $\overline{A}^\omega$ is $\overline{A}$-codense).*

1. *If $\overline{D}$ is $\overline{A}$-codense and $\overline{E}$ is $\overline{A}$-dense, then $\overline{D} \to \overline{E}$ is $\overline{A}$-dense.*
2. *If $\overline{D}$ is dense and $\overline{E}$ is $\overline{A}$-codense, then $\overline{D} \to \overline{E}$ is $\overline{A}$-codense.*

Proof. 1. By assumption there is a computable separating and total $f \in D \to A^\omega$, and $\overline{A}^\omega \to \overline{E}$ is dense. We have to show that $\overline{A}^\omega \to \overline{D} \to \overline{E}$ is dense in $A^\omega \to D \to E$. Via isomorphism it suffice to show that $\overline{A}^\omega \times \overline{D} \to \overline{E}$ is dense in $A^\omega \times D \to E$. Let h_0 be a compact element of $A^\omega \times D \to E$. We have to construct some total $h \in A^\omega \times D \to E$ extending h_0. Since h_0 is compact there are finitely many compacts $a_i \in A_0$, $d_i \in D_0$, $e_i \in E_0$ ($i \in I$, I finite) such that

$$h_0(a,d) = \bigsqcup\{e_i \mid i \in I,\ a_i \sqsubseteq a,\ d_i \sqsubseteq d\}$$

for all $(a,d) \in A \times D$. Define a function pair $\in A \times A \to A$ by $\mathrm{pair}(a,b) := \mathrm{melt}(\lambda n.\mathbf{if}\, n = 0\, \mathbf{then}\, a\, \mathbf{else}\, b)$. Clearly pair is computable, total and separating. Since also f is separating we have for all $i, j \in I$

$$(a_i, d_i) \uparrow (a_j, d_j) \Leftrightarrow \mathrm{pair}(a_i, f(d_i)) \uparrow \mathrm{pair}(a_j, f(d_j))$$

By algebraicity of A there are compacts $b_i \sqsubseteq \mathrm{pair}(a_i, f(d_i))$ ($i \in I$) such that for all $i, j \in I$

$$(a_i, d_i) \uparrow (a_j, d_j) \Leftrightarrow b_i \uparrow b_j$$

Hence the function $g\colon A \to E$

$$g_0(a) := \bigsqcup\{e_i \mid i \in I,\ b_i \sqsubseteq a\}.$$

is a well-defined compact in $A \to E$. By assumption there is a total $g \in A \to E$ extending g_0. Now define $h\colon A \times D \to E$ by

$$h(a,d) := g(\mathrm{pair}(a, f(d)))$$

Obviously h is continuous and total. For $i \in I$ we have

$$h(a_i, d_i) = g(\mathrm{pair}(a_i, f(d_i))) \sqsupseteq (a_i) \sqsupseteq e_i.$$

Hence h extends h_0.

2. By assumption $\overline{D}$ is dense which means that there is a computable total $s \in D^\omega$ such that $s[\mathbf{N}]$ is dense in D. Furthermore we have some computable separating and total function $f \in E \to A$. Define $g \in (D \to E) \to A$ by

$$g(h) := \mathrm{melt}(f \circ h \circ s)$$

Clearly g is computable and total. To see that g is separating let $h_0 \not\uparrow h_1$ in $D \to E$. Since $s[\mathbf{N}]$ is dense in D there is n such that $h_0(s(n)) \not\uparrow h_1(s(n))$. Hence $f(h_0(s(n))) \not\uparrow f(h_1(s(n)))$ in A because f is separating. It follows that $f \circ h_0 \circ s \not\uparrow f \circ h_1 \circ s$. Therefore $g(h_0) \not\uparrow g(h_1)$, since melt is separating. □

By the facts 1., 2. and 3. listed above this theorem generalises the density theorem in [3]. Note also that the witnesses of density and the separating functions in the conclusions of the theorem are defined explicitely from the corresponing objects given by the assumtions using just case analysis on $n = 0$, $n > 0$. Since obviously $\overline{D}$ is $\overline{D}^\omega$-codense for every domain with totality $(D, \overline{D}$. we have the following corollary.

Theorem 8 *Let $(D, \overline{D})$ be a domain with strong totality such that $\overline{D}$ is nonempty and D^ω-dense. Then the total continuous functionals $\overline{D}_\rho$ and also the hereditarily computable total continuous functionals over $\overline{D}$ are domains with strong dense totalities.*

Moreover for every type ρ a dense and total sequence in D_ρ can be defined *explicitely from a computable dense and total sequence in $D^\omega \to D$, case analysis on $n = 0$, $n > 0$, and a bijection from $\boldsymbol{N}^2$ to ω.*

In [33] it has been shown for domain $(R, \overline{R})$ of partial and total reals that $\overline{R}^n \to \overline{R}$ is dense uniformly for all n. Obviously this implies that $\overline{R}$ is $\overline{R}^\omega$-dense. Hence we obtain Normann's density theorem in [33].

Theorem 9 *The total continuous functionals over the reals, $\overline{R}_\rho$, are dense in R_ρ for all finite types ρ.*

Closing this section we state a simple but important application of density [23,39,3]:

Theorem 10 (Effective choice principle) *Let $(D_\rho)_\rho$ be the hierarchy of partial continuous functional over the integers. For all types ρ and σ there is a PCF+POR definable total functional of type $(\rho \times \sigma \to 0) \to (\rho \to \sigma)$ computing for every total functional f of type $\rho \times \sigma \to 0$ such that*

$$\forall x \in \overline{D}_\rho \exists y \in \overline{D}_\sigma \; f(x, y) = 0$$

a total functional g of type $\rho \to \sigma$ such that

$$\forall x \in \overline{D}_\rho \; f(x, g(x)) = 0.$$

4 Computability and totality

Now we use the results of the previous section to prove the equivalence of the computability concepts 1. 2. and 3. for total continuous functionals.

In [17] it had been proven that the fan-functional computing a modulus of uniform continuity of a total type-2 functional restricted to a compact fan is (S1-S9)-definable (see also [30]), but in [3] it had been shown that the fan-functional *is* (S1-S9) computable if Kleene's schemata (S1-S9) are interpreted in the partial continuous functionals. It was then conjectured that *every* computable total continuous functional over the integers is (S1-S9)-computable, i.e. PCF-definable (see theorem 4). Again it was Normann [34] who proved in 1998 this conjecture thus showing the computation concepts 1. and 2. to be equivalent for the total continuous functionals.

Theorem 11 *Every computable total continuous functional over the integers is PCF-definable.*

Moreover for every type ρ there is a PCF-computable functional of type $(0 \to 0) \to \rho$ computing from every enumeration of the compact approximations of total functional f of type ρ (where this enumeration is coded as sequence of integers) a total functional $\hat{f} \sqsubseteq f$

Normann's proof uses the density theorem in an essential way.

In order to prove the equivalence of the computation principles 2. and 3. we have to look for a total analogon of the generalized Myhill-Shepherdson Theorem (theorem 2). In [3] several such theorems are proved which may be viewed as a generalization of the *Kreisel-Lacombe-Shoenfield Theorem* [24]. Here, we only present one of them.

An element y of a domain E is called *almost maximal* if it cannot be extended in two inconsistent ways, i.e. $\forall y', y''(y \sqsubseteq y', y'' \Rightarrow y' \uparrow y'')$. Using the axiom of choice this can be shown to be equivalent with the property that y has precisely one maximal extension (but we will not use this fact). For instance, the elements of a co-dense set are almost maximal. Also all elements of $\overline{R}$ are almost maximal (although $\overline{R}$ is not a co-dense set).

Theorem 12 *Let D, E be effective domains with totality. Assume that $\overline{D}$ is effectively dense and all elements of $\overline{E}$ are almost maximal. Then every effective operation $f\colon \overline{D} \to \overline{E}$ can be extended to an effective (and by the generalized Myhill-Shepherdson Theorem continuous) operation $f'\colon D \to E$, in the sense that $f(x) \sqsubseteq f'(x)$ for all $x \in \overline{D}$.*

From this theorem we may deduce the equivalence of the computation concepts 2. and 3. for many instances of total functions. For instance it immediately entails the well-known theorem of Ceitin and Moschovakis saying that every effective operator on the reals is continuous. We will see that it also implies the equivalence of the hereditarily computable total continuous functionals and hereditarily effective total operations of finite types over a large class of base types.

To make this precise we need the notion of a partially numbered set introduced by Ershov. A *partially numbered set* is a pair (S, ν) where A is a set and ν is a surjection from a subset $\delta\nu \subseteq \mathbf{N}$ onto S [12]. For example if $\overline{D}$ is a strong totality on a domain D such that $\overline{D} \subseteq D_{\text{comp}}$ then any principal constructivation ν of D induces a partial numbering $\nu\colon \nu^{-1}\overline{D} \to \mathbf{D}$ defined by $\nu n := [\nu n]$. An *effective operation* between two numbered sets (S, ν), (T, μ) is a mapping $F\colon S \to T$ that is tracked by some partial recursive function f i.e. f is defined on $\delta\nu$, $f[\delta\nu] \subseteq \delta\mu$ and $F(\nu n) = \mu f(n)$ for all $n \in \delta\nu$. By $\mathbf{EO}(S, T)$ we denote the set of effective operations from T to S partially numbered by Kleene indices of tracking functions. Obviously this corresponds precisely to the exponential in the category PER of partial equivalence relations. Given a partially numbered set S as base type we define the *effective operations of finite types over S* by

$$S_\tau := S, \quad S_{\rho\to\sigma} := \mathbf{EO}(S_\rho, S_\sigma).$$

Starting with the base type $\mathbf{N}$ numbered by the identity we obtain the hereditarily effective operations [44]. If we start with $\mathbf{N}_\perp$ numbered by a principal constructivation we obtain the hereditarily partial effective operations (see Theorem 3). From theorems 8 and 12 we can now easily derive the following theorem.

Theorem 13 *Let D be a domain and set*

$$\overline{D} := \{x \in D \mid x \textit{ computable and almost maximal}\}.$$

Assume that $\overline{D}^\omega \to \overline{D}$ is dense and that there is a total computable function *select* $\in D \to D$ *such that* $\forall x, y \in \overline{D}(x \uparrow y \Rightarrow x \uparrow select(x) = select(y))$.

Then the hereditarily effective operations of finite type over $\overline{D}$ *are effectively isomorphic with the hereditarily computable total continuous functionals over the* $\overline{D}$.

The theorem applies for instance to $\overline{D} := \mathbf{N}$ and to $\overline{D} := \overline{R}$.

5 Dependent domains and universes

In this paper we focused on the type constructor $\to$ (function space). However some of the work described has been extended to dependent products, dependent sums and universe operators in the sense of Martin-Löf type theory [35,31,25,45,4].

Palmgren and Stoltenberg-Hansen [35] developed the notion of a dependent domain and gave a domain interpretation a partial type theory. Kristiansen and Normann [31,25] used a universe of dependent domains with dense totality to represent computations relative to certain noncontinuous functionals like ^3E. Waagbø modified Palmgren's and Stoltenberg-Hansen's work for interpreting (the usual) total type theory using dependent domains with totality. In [4] abstract density theorems for dependent types and universe operators are proved. This has been used in [45] for proving the equivalence of the computation concepts 2. and 3. for functionals of dependent types over $\mathbf{N}$.

Theorem 14 *Normann's well-founded hierarchy of hereditarily computable total continuous functionals of dependent types and Beeson's model of total effective operations of dependent types [1] are effectively isomorphic.*

6 Related work

There exists a substantial literature discussing different notions of computability for higher types, e.g. [36,27,29,17,16,19,30,11]. Much of this work focuses on Kleene's schemata (S1-S9) [20] which, when interpreted on the *partial* continuous functionals, are equivalent to PCF as shown by Platek [36,27]. In combination with Normann's result mentioned above this implies that for the total continuous functionals recursive continuity and (S1-S9) interpreted on the partial continuous functionals coincide. This seems to contradict earlier results showing that (S1-S9) computability is strictly weaker than recursive continuity (cf. e.g. [30]). In fact however it just shows that the two interpretations of (S1-S9) give rise to different notions of computability on the total continuous functionals. This answers a question posed by Cook in [11], p. 59.

Cook's paper also gives an introduction into feasibility for higher type functionals, a subject which is not touched here, since our concepts 1. 2.,and 3. are all Turing complete in the sense that they define exactly those (partial) number theoretic functions (represented by strict functions of type level one) that are (partial) recursive. There are versions of 2. that induce a notion of subrecursiveness or feasibility in higher type (e.g. [10,?]), but I don't know of any convincing approach defining subrecursiveness in higher types via restrictions of the concepts 1. or 3.

In [38] it is shown how the equivalence of 1. and 2. can be used to prove full abstraction results for functional languages w.r.t. the continuous denotational semantics. In [6] Ershov's method of defining the total continuous functionals within domain theory is generalised and used to define effective domain representations for large classes of topological spaces. In [7] this theory is applied to an analysis of continuous stream transformers. [43] compares effective domain representations of topological algebras with other approaches to computability on toplogical spaces. The equivalence of the concepts 2. and 3. for effective metric spaces first proven by Ceitin [8] and Moschovakis [28] has been generalized to effective topological spaces in [42].

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