Concurrent and nondeterministic program extraction

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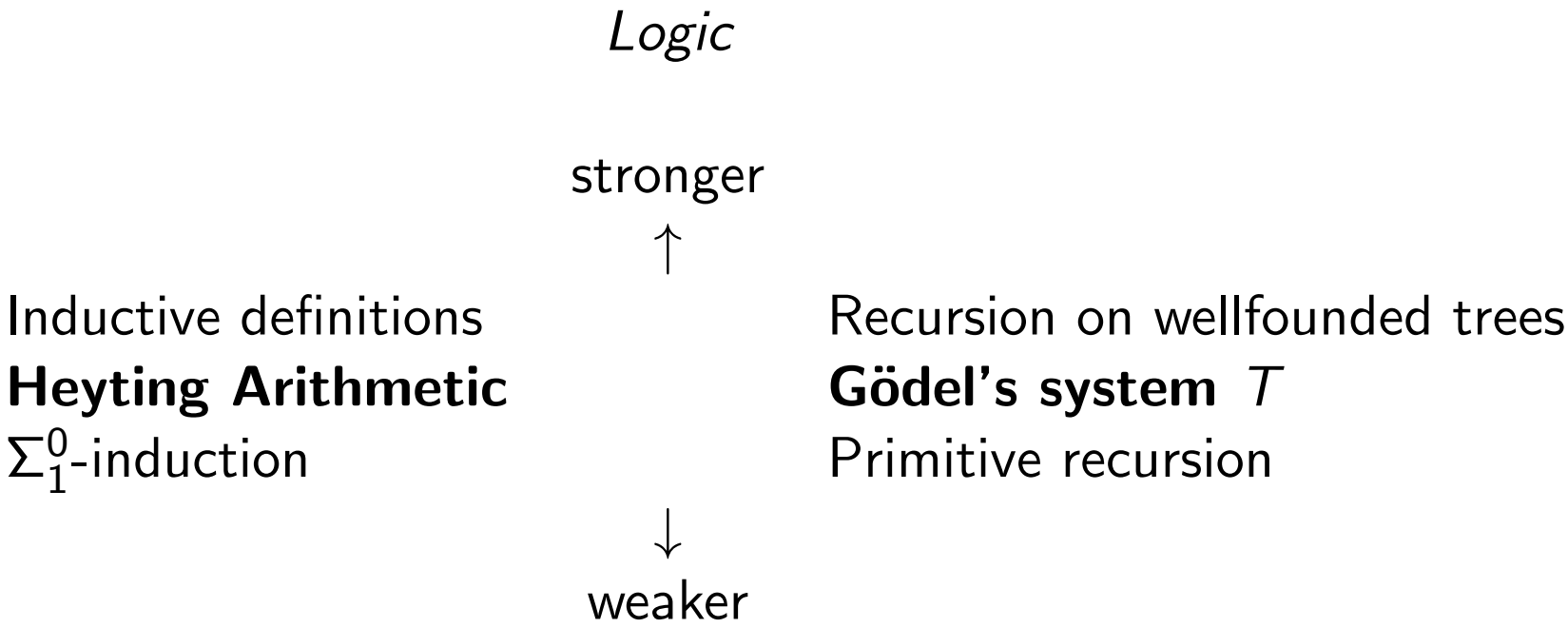
Proofs as Programs



Proofs as Programs

Logic

stronger

↑

Inductive definitions	Recursion on wellfounded trees
Heyting Arithmetic	**Gödel's system** T
Σ^0_1-induction	Primitive recursion

↓

weaker

CS

Classical logic	Non-local control
?	Side effects
?	Concurrency, nondeterminism

Intuitionistic Fixed Point Logic (IFP)

- Intuitionistic many-sorted logic in finite types.
- Sorts represent (carriers of) abstract mathematical structures given by *non-computational (nc)*, that is, disjunction-free axioms.
- Inductive and coinductive definitions of predicates as least and greatest fixed points of monotone predicate transformers.

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Our leading example will be from computable analysis.

Realizability

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The definition of realizability is as usual (Kleene) except that quantifiers are interpreted uniformly:

- $a \,\mathbf{r}\, \exists x\, A(x)$ means $\exists x\, (a \,\mathbf{r}\, A(x))$.
- $a \,\mathbf{r}\, \forall x\, A(x)$ means $\forall x\, (a \,\mathbf{r}\, A(x))$.

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- $a \,\mathbf{r}\, \forall x\, A(x)$ means $\forall x\, (a \,\mathbf{r}\, A(x))$.

A realizer of an implication $A \to B$ is a function f such that whenever a realizes A, then $f(a)$ realizes B.

f may be partial, i.e. $f(a)$ need not be defined for all a.

Realizability involving nc formulas

If A is nc, then $a\,\mathbf{r}\,A$ means just A.

Hence, if A and B are nc formulas, then

$$a\,\mathbf{r}\,(A \vee B) \text{ means } (a = 0 \wedge A) \vee (a = 1 \wedge B)$$

If A is nc, then $a\,\mathbf{r}\,(A \to C)$ means $A \to a\,\mathbf{r}\,C$.

Real, natural and rational numbers

The structure of the *real numbers* $\mathbb{R} = (0, 1, +, *, -, /, <, |\cdot|)$ is treated as a sort with constants and functions specified by nc axioms.

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The *natural numbers* are defined as the least subset of $\mathbb{R}$ that contains 0 and is closed under successor:

$$\mathbb{N} \stackrel{\mu}{=} \{\, 0 \,\} \cup \{\, x + 1 \mid x \in \mathbb{N} \,\}$$

Realizability automatically associates with this definition the unary representation of natural numbers and with proofs of closure properties of $\mathbb{N}$ programs operating on that representation.

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$$\mathbb{Q} := \{\, p * \frac{m}{n} \mid p \in \{-1, 1\}, m, n \in \mathbb{N}, n \neq 0 \,\} \subseteq \mathbb{R}$$

Cauchy and Signed Digit representation

A **Cauchy representation** of a real number $x \in \mathbb{I} = [-1, 1] \subseteq \mathbb{R}$ is a sequence $f : \mathbb{N} \to \mathbb{Q}$ such that for all $n \in \mathbb{N}$

$$|x - f(n)| \leq 2^{-n}$$

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$$|x - f(n)| \leq 2^{-n}$$

A **signed digit representation** of a real number $x \in \mathbb{I}$ is a stream $d_0 : d_1 : \ldots \in \mathrm{SD}^\omega$, where $\mathrm{SD} = \{-1, 0, 1\}$, such that

$$x = \sum_{i \in \mathbb{N}} d_i * 2^{-(i+1)}$$

Gray code

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Tsuiki extended this to a representation of real numbers.

Hideki Tsuiki: Real Number Computation through Gray Code Embedding. TCS 284, 2002.

Tsuiki's partial infinite Gray code for real numbers

The **Gray code** of $x \in [-1, 1]$ is the itinerary of the *tent map*

$$\mathrm{t}(x) = 1 - 2|x|$$

This means that the n-th digit is 0 resp. 1 if $\mathrm{t}^n(x) < 0$ resp. > 0.

If $\mathrm{t}^n(x) = 0$, then the n-th digit is undefined.

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Remarkably, **every real in** $[-1, 1]$ **has a unique Gray code**.

Since $\mathrm{t}(0) = 1$ and $\mathrm{t}(1) = \mathrm{t}(-1) = -1$, at most one digit of the Gray code can be undefined. Therefore, computation with the Gray code can be modeled by a *Two-Head-Turing-Machine*.

Translating Gray code to signed digit representation

Tsuiki's Two-Head Turing Machines can execute the following translation of Gray code to signed digit representation.

```
gtos (0:s)     = -1 : gtos s
gtos (1:b:s)   =  1 : gtos (swap b : s)
gtos (a:1:c:s) =  0 : gtos (a : swap c : s)
```

Such a machine cannot be extracted from a proof in the current system since it exhibits a kind of concurrency or nondeterminism, that is absent in extracted programs.

Problem: Devise a logic that can extract concurrent nondeterministic programs.

Related work: Realizing Gray Code deterministically

The closest to Gray code one seems to get with traditional program extraction is a program that works on so-called pre Gray code, i.e., streams representing constructions of Gray code.

B., Kenji Miyamoto, Helmut Schwichtenberg, Hideki Tsuiki: Logic for Gray-code computation (accepted for publication)

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The closest to Gray code one seems to get with traditional program extraction is a program that works on so-called pre Gray code, i.e., streams representing constructions of Gray code.

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This approach is currently being extended to pre-Gray code for compact sets by Dieter Spreen and Hideki Tsuiki.

Cauchy representation in logical form

Define $\mathrm{A}(x) := \forall n \in \mathbb{N}\, \exists q \in \mathbb{Q} \cap \mathbb{I}\,.\, |x - q| \leq 2^{-n}$

Then $f \mathbf{r} A(x)$ iff f is a Cauchy representation of x.

We call the predicate $A(x)$ the *Cauchy representation in logical form*.

Signed digit representation in logical form

$$\mathrm{C}(x) \stackrel{\nu}{=} \exists d \in \mathrm{SD} \,.\, x \in \mathbb{I}_d \wedge \mathrm{C}(2x - d)$$

where $\mathbb{I}_d := [d/2 - 1/2, d/2 + 1/2]$ and $\stackrel{\nu}{=}$ means 'largest'.

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A realizer of $\mathrm{C}(x)$ is a stream of signed digits whose head realizes $x \in \mathbb{I}_d$ for some $d \in \mathrm{SD}$ (which essentially means the head equals d) and whose tail realizes $\mathrm{C}(2x - d)$.

It follows that the realizers of $\mathrm{C}(x)$ are exactly the signed digit representations of x.

Gray code in logical form

$$\mathrm{G}(x) \overset{\nu}{=} \mathrm{D}(x) \wedge \mathrm{G}(\mathrm{t}(x))$$

where $\mathrm{D}(x) := x \neq 0 \rightarrow x \leq 0 \vee x \geq 0$.

Gray code in logical form

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where $\mathrm{D}(x) := x \neq 0 \to x \leq 0 \vee x \geq 0$.

Note that $a \,\mathbf{r}\, D(x)$ means
$x \neq 0 \to (a = 0 \wedge x \leq 0) \vee (a = 1 \wedge x \geq 0)$.

Hence a realizer of $\mathrm{G}(x)$ is a stream of partial binary digits whose head realizes $\mathrm{D}(x)$ and whose tail realizes $\mathrm{G}(\mathrm{t}(x))$.

This means that the Gray code of x is a realizer of $\mathrm{G}(x)$.

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It is our goal to show constructively $\mathrm{A} = \mathrm{C} = \mathrm{G}$ which will give us programs translating between the three representations.

We concentrate on the inclusion $\mathrm{G} \subseteq \mathrm{C}$.

Proving $\mathrm{G} \subseteq \mathrm{C}$

The only really hard part of the proof is to determine the first signed digit of an $x \in \mathrm{G}$, that is, to show

(*) If $\mathrm{G}(x)$, then $x \in \mathbb{I}_d$ for some $d \in \mathrm{SD}$.

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We use the following *Disjunction Principle*:

(DP) $(A \overset{P}{\vee} B) \wedge (P \overset{Q}{\vee} C) \to (A \vee B) \vee C$

where

$$A \overset{P}{\vee} B := (P \to A \vee B) \wedge (\neg P \to A \wedge B)$$

and A, B, C, P, Q range over nc propositions.

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We apply (DP) with

$$A := x \in \mathbb{I}_{-1}, \quad B := x \in \mathbb{I}_1, \quad C := x \in \mathbb{I}_0,$$
$$P := x \neq 0, \quad Q := \mathrm{t}(x) \neq 0.$$

Concurrent Fixed Point Logic

DP is classically trivial, but constructively invalid, in particular *not realizable*.

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Concurrent Fixed Point Logic (CFP) comes to our rescue.

We add a modal operator S and the rules

$$\frac{\Gamma \vdash A}{\Gamma \vdash \mathrm{S}(A)}\ (\mathrm{S}^+) \qquad \frac{\Gamma \vdash \mathrm{S}(A) \qquad \Gamma, A \vdash \mathrm{S}(B)}{\Gamma \vdash \mathrm{S}(B)}\ (\mathrm{S}^-)$$

Realizing $\mathrm{S}(A)$

A realizer of $\mathrm{S}(A)$ is a partial family a, indexed by some discrete set $\mathcal{I}$, such that

(i) $a(i)$ yields a result for at least one $i \in \mathcal{I}$,

(ii) for any $i \in \mathcal{I}$, if $a(i)$ yields a result b, then b realizes A.

Embedding IFP

To each IFP formula A we assign a CFP formula A^{S} by applying to each disjunctive and existential subformula the modality S.

Theorem (Embedding)

If $\Gamma \vdash_{\mathrm{IFP}} A$, *then* $\Gamma^{\mathrm{S}} \vdash_{\mathrm{CFP}} A^{\mathrm{S}}$.

Theorem (Soundness for CFP)

From a proof in CFP *of* $\Delta, \Gamma \vdash A$, *where* Δ *consists of nc formulas, one can extract a concurrent program term* M *such that* $\Delta, \vec{a}\,\mathbf{r}\,\Gamma \vdash (M\,\vec{a})\,\mathbf{r}\,A$.

Theorem (Concurrent Soundness)

From a proof of $\Delta, \Gamma \vdash_{\mathrm{IFP}} A$, *where* Δ *is nc, one can extract a concurrent program* M *such that* $\Delta, \vec{a}\,\mathbf{r}\,\Gamma^{\mathrm{S}} \vdash (M \cdot \vec{a})\,\mathbf{r}\,A^{\mathrm{S}}$.

Concurrently realizing the Disjunction Principle

Note that DP^{S} is (equivalent to)

$$(A \overset{P}{\vee} B)^{\mathrm{S}} \wedge (P \overset{Q}{\vee} C)^{\mathrm{S}} \to \mathrm{S}(A \vee B \vee C)$$

where

$(A \overset{P}{\vee} B)^{\mathrm{S}} = (P \to \mathrm{S}(A \vee B)) \wedge (\neg P \to A \wedge B)$,

$(P \overset{Q}{\vee} C)^{\mathrm{S}} = (Q \to \mathrm{S}(P \vee C)) \wedge (\neg Q \to P \wedge C)$

The following function realizes DP^{S}

```
fDP (0,b) = -1
fDP (1,b) = 1
fDP (a,1) = 0
```

These equation must be interpreted as nondeterministic rewrite rules, similar to the program `gtos`.

Extracted programs for Gray code

Lemma 9. If $x \in \mathrm{G}$, then $x \in \mathbb{I}_d$ for some $d \in \mathrm{SD}$.

```
f9 (a:s) = conv a
f9 (a:1:s) = 0

  where conv 0 = -1
        conv 1 = 1
```

The equations above should be read as overlapping reduction rules.

Lemma 10. If $x \in \mathrm{G}$, then $-x \in \mathrm{G}$.

```
f10 (a:s) = swap a : s
```

Extracted programs for Gray code ctd.

Lemma 11. If $0 \leq x \leq 1$ and $\mathrm{G}(x)$, then $\mathrm{G}(2x - 1)$.

```
f11 (a:s) = f10 s
```

hence

```
f11 (a:b:s) = swap b : s
```

Lemma 12. If $-1 \leq x \leq 0$ and $\mathrm{G}(x)$, then $\mathrm{G}(2x + 1)$.

```
f12 (a:s) = s
```

Extracted programs for Gray code ctd.

Lemma 13. If $0 \leq x \leq 1$ and $\mathrm{G}(x)$, then $\mathrm{G}(1-x)$.

```
f13 (a:s) = 1 : f11 (a:s)
```

Hence

```
f13 (a:b:s) = 1 : swap b : s
```

Lemma 14. If $-\frac{1}{2} \leq x \leq \frac{1}{2}$ and $\mathrm{G}(x)$, then $\mathrm{G}(2x)$.

```
f14 (a:s) = a : f13 s
```

Hence

```
f14 (a:b:c:s) = a : swap c : s
```

Extracted programs for Gray code ctd.

Lemma 15. $\mathrm{G} \subseteq \mathrm{C}$.

```
f15 s = let { d = f9 s}
        in d : case d of { -1 -> f12 s ;
                           0 -> f14 s ;
                           1 -> f11 s }
```

Hence

```
f15 (0:s)     = -1 : f15 s
f15 (1:a:s)   =  1 : f15 (swap a : s)
f15 (a:1:c:s) =  0 : f15 (a : swap c : s)
```

Again, read the equations above as overlapping rewrite rules.

One observes that the equations for `f15` correspond exactly to those given by Tsuiki.

Parallelism in Exact Real Number Computation

- Computing with the interval domain as a model of real numbers appears to require a parallel if-then-else operation (Potts, Edalat, Escardo, 1997).
- In fact, this parallelism is unavoidable (Escardo, Hofmann, Streicher, 2004).
- Computing with TTE representations (e.g. Cauchy- or signed digit representation) does *not* require parallelism.
- Gray code (though very similar to signed digits) requires nondeterministic parallelism.

Nondeterministic realizability

This talks was based on the paper

- B. Extracting nondeterministic concurrent programs (accepted for CSL 2016)

which considers realizing program language with an infinite choice construct, $\langle\alpha\rangle M$, and the reduction rules

$$\langle\alpha\rangle M \longrightarrow M[i/\alpha] \qquad (i \in \mathcal{I})$$

where $\mathcal{I}$ is an infinite index set.

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$$\langle\alpha\rangle M \longrightarrow M[i/\alpha] \qquad (i \in \mathcal{I})$$

where $\mathcal{I}$ is an infinite index set.

This can be reduced to binary choice

$$M + N \longrightarrow M, \qquad M + N \longrightarrow N$$

as in

- Bucciarelly, Ehrhard, Manzonetto. A relational semantics for parallelism and non-determinism in a functional setting. APAL 163(7): 918-934 (2012)

Thank You

Determining the first signed digit of an $x \in \mathrm{G}$

Lemma

If $\mathrm{G}(x)$, *then* $x \in \mathbb{I}_d$ *for some* $d \in \mathrm{SD}$.

Proof.

Assume $\mathrm{G}(x)$. Then $\mathrm{D}(x)$ and $\mathrm{D}(\mathrm{t}(x))$. Apply (DP) with

$$A := x \in \mathbb{I}_{-1}, \quad B := x \in \mathbb{I}_1, \quad C := x \in \mathbb{I}_0,$$

$$P := x \neq 0, \quad Q := \mathrm{t}(x) \neq 0.$$

We have to show $A \overset{P}{\vee} B$ and $P \overset{Q}{\vee} C$.

- $P \to A \vee B$ is $\mathrm{D}(x)$.
- To show $\neg P \to A \wedge B$, assume $\neg(x \neq 0)$. Then clearly $x \in \mathbb{I}_{-1}$ and $x \in \mathbb{I}_1$.
- To show $Q \to P \vee C$, assume $\mathrm{t}(x) \neq 0$. Then $\mathrm{t}(x) \leq 0 \vee \mathrm{t}(x) \geq 0$, since $\mathrm{D}(\mathrm{t}(x))$. If $\mathrm{t}(x) \leq 0$, then $|x| \geq \frac{1}{2}$, hence $x \neq 0$. If $\mathrm{t}(x) \geq 0$, then $x \in \mathbb{I}_0$.
- To show $\neg Q \to P \wedge C$, assume $\neg(\mathrm{t}(x) \neq 0)$. Then $|x| = \frac{1}{2}$, hence $x \neq 0$ and $x \in \mathbb{I}_0$.