Program extraction in computable analysis

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Computability in Europe 2011

Sofia, Bulgaria

Outline

The Curry-Howard correspondence (or Brouwer-Heyting-Kolmogorov interpretation)

Formulas correspond to *data types*
Proofs correspond to *programs*

$A \vee B$	disjoint sum
$A \wedge B$	cartesian product
$A \to B$	function space
$\exists x\, A$	(dependent) cartesian product
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A proof of a formula A corresponds to a program constructing an element of A.

- What is a function?
- What if the quantified x ranges over abstract objects?
- How do we interpret logical axioms, e.g. $A \vee \neg A$?
- How do we interpret maths axioms, e.g. induction, choice?
- Why is it interesting and useful?

Why Curry-Howard is interesting and useful

Foundations

Constructive foundation of Mathematics (Brouwer, Heyting, Kolmogorov, Gödel, Kleene, Kreisel, Martin-Löf). Properties of logical and mathematical systems (Realizability $\Rightarrow$ existence and disjunction property; Dialectica Interpretation $\Rightarrow$ consistency)

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Mathematics

Approximation-, fixedpoint-, ergodic-theory (Kohlenbach, Avigad, . . . , using DI). The study of function spaces led to new developments in computability theory, topology, domain theory. The problem of C-H interpreting classical choice axioms has led to new recursion principles such as bar recursion and products of selection functions (see recent work by Martin Escardo and Paulo Oliva).

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- f should be computable. What does this mean if A itself consists of functions? ($\Rightarrow$ computability in higher types)
- Don't we need a *proof* that f does it's job? (circularity!)

Realizing an implication

Realizability (Kleene, Kreisel)

$$f\,\mathbf{r}\,(A \to B) \quad \equiv \quad \forall a\,(a\,\mathbf{r}\,A \to f(a)\,\mathbf{r}\,B)$$

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$$(f,g)\,\mathbf{r}\,(A \to B) \quad \equiv \quad \forall a, v\,(a\,\mathbf{r}_{g(a,v)}\,A \to f(a)\,\mathbf{r}_v\,B)$$

where $a\,\mathbf{r}\,A \equiv \forall u\,(a\,\mathbf{r}_u\,A)$ and $b\,\mathbf{r}\,B \equiv \forall v\,(b\,\mathbf{r}_v\,B)$ are purely universal formulas.

The idea is that for the conclusion, $f(a)\,\mathbf{r}_v\,B$, the premise, $\forall u\,(a\,\mathbf{r}_u\,A)$ is used for finitely many u only (continuity argument), in fact, a single $u = g(a,v)$ suffices.

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Both interpretations extract from a proof of A a term M and a proof of $M\,\mathbf{r}\,A$ (Soundness Theorem).

In the DI the proof of $M\,\mathbf{r}\,A$ takes place in a quantifier free system!

Realizing quantifiers

Traditionally:

$$\begin{aligned} (x, a)\,\mathbf{r}\,\exists x\,A(x) &\equiv a\,\mathbf{r}\,A(x) \\ f\,\mathbf{r}\,\forall x\,A(x) &\equiv \forall x\,(f(x)\,\mathbf{r}\,A(x)) \end{aligned}$$

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Alternative: uniform realization of quantifiers

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For concrete objects we may relativize the quantifiers:

$$\forall x\,(\mathbb{N}(x) \to \exists y\,(\mathbb{N}(y) \wedge (x = 2y \vee x = 2y + 1)))$$

where $\mathbb{N}$ is defined such that $n\,\mathbf{r}\,\mathbb{N}(x)$ means that n is a representation of the natural number x.
The extracted program computes integer division by 2.

Program extraction and the law of excluded middle

Realizing, say, $\forall x\,(\mathbb{N}(x) \to A(x) \vee \neg A(x))$ would mean to construct a program computing for every (representation of) a natural number x a realizer of $A(x)$ or a realizer or $\neg A(x)$. This is impossible, in general.

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But, one can eliminate LEM in proofs of formulas of the form

$$\forall x\,(\mathbb{N}(x) \to \exists y\,(\mathbb{N}(y) \wedge A_0(x, y))$$

where $A_0(x, y)$ is decidable, using Gödel's negative translation and the Friedman/Dragalin A-translation.

Other approaches to program extraction from classical proofs

- ϵ-substitution calculus (Hilbert).
- Interpretation of $\neg\neg A \to A$ by continuations (Felleisen).
- Direct computational interpretation of classical sequent calculus ($\lambda\mu$-calculus, Parigot).
- Interpretation of restricted forms of LEM by learning based realizability (Berardi, Aschieri)
- Realizability interpretation of classical systems via stacks and processes (Krivine).

Interpreting induction

Induction on natural numbers

$$A(0) \wedge \forall x\,(A(x) \to A(x+1)) \to \forall x\,(\mathbb{N}(x) \to A(x))$$

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In general, one has for a monotone predicate transformer Φ an induction schema for its least fixed point $\mu\Phi$:

$$\Phi(\mathcal{P}) \subseteq \mathcal{P} \to \mu\Phi \subseteq \mathcal{P}$$

The data type associated with $\mu\Phi$ is the initial algebra $\mathrm{In}_\varphi : \varphi(\mu\varphi) \to \mu\varphi$ of a functor φ derived from Φ. The induction scheme is realized by the iterator $\mathbf{It}_\varphi$ that iterates any "step function" (i.e. φ-algebra) $f : \varphi(\alpha) \to \alpha$ to an algebra morphism $\mathbf{It}_\varphi(f) : \mu\varphi \to \alpha$ with computation rule (i.e. morphism equation)

$$\mathbf{It}_\varphi(f)\,\mathrm{In}_\varphi(m) = f(\mathbf{map}_\varphi(\mathbf{It}_\varphi(f))(m))$$

Example: Natural numbers

Recall $\mathbb{N} = \mu\Phi$ where

$$\begin{aligned} \Phi(X) &= \{0\} \cup \{x+1 \mid x \in X\} \\ &= \{y \mid y = 0 \vee \exists x\,(y = x+1 \wedge x \in X)\} \end{aligned}$$

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The functor associated with Φ is obtained by removing all first-order parts from Φ: $\varphi(\alpha) = 1 + \alpha$. The initial algebra $\mathrm{In}_\varphi : \varphi(\mu\varphi) \to \mu\varphi$ is the familiar structure of unary natural numbers $\mathbb{N} := \mu\varphi$ generated by zero and successor.

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A step function $f : \varphi(\alpha) \to \alpha$ consists of $f_0 : \alpha$ and $f_1 : \alpha \to \alpha$. The iteration $g := \mathbf{It}_\varphi(f) : \mathbb{N} \to \alpha$ is defined recursively by $g(0) = f_0$, $g(\mathrm{S}(n)) = f_1(g(n))$.

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Remarks: 1. The variables x, y may range over abstract objects, for example the real numbers. 2. Category theory is only needed to explain realizability. The "user" doesn't have to know anything about this.

Interpreting choice axioms

The (constructive) axiom of choice

$$\forall x\, \exists y\, A(x, y) \to \exists f\, \forall x\, A(x, f(x))$$

has a trivial realizer, namely the identity (both with the traditional and the uniform interpretation of quantifiers)

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Much harder is the *classical* axiom of choice which is obtained by double negation translation of the constructive axiom of choice. Even classical *countable* classical choice is hard to realize:

$$\forall x \in \mathbb{N}\, (\neg\neg \exists y\, A^{\neg\neg}(x, y) \to \neg\neg \exists f\, \forall x\, A^{\neg\neg}(x, f(x)))$$

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In contrast, the negative translation of an induction axiom is unproblematic, since it is again instances of an induction axiom.

Reals as processes

We view a real number x as a *process* that emits digits providing better and better approximations to x.

Processes are conveniently modelled by *final coalgebras*.

Realizability naturally associates final coalgebras with *coinductive definitions*, i.e. greatest fixed points of monotone predicate transformers (in the same way as it associates initial algebras with inductive definitions.

Hence, we use coinductive definitions to model a digital approach to computable analysis.

Coinduction

Coinduction is dual to induction. Given a monotone predicate transformer Φ we have a coinduction scheme for its greatest fixed point $\nu\Phi$:

$$\mathcal{P} \subseteq \Phi(\mathcal{P}) \to \mathcal{P} \subseteq \nu\Phi$$

The associated data type is the final coalgebra $\mathrm{Out}_\varphi : \nu\varphi \to \varphi(\mu\varphi)$.

The coinduction scheme is realized by the coiterator $\mathbf{Coit}_\varphi$ that coiterates any "step function" (i.e. φ-coalgebra) $f : \alpha \to \varphi(\alpha)$ to a coalgebra morphism $\mathbf{Coit}_\varphi(f) : \alpha \to \mu\varphi$ with computation rule (i.e. morphism equation)

$$\mathrm{Out}_\varphi(\mathbf{Coit}_\varphi(f)(a)) = \mathbf{map}_\varphi(\mathbf{Coit}_\varphi(f))(f(a))$$

Equivalently, using the fact that Out_φ has an inverse In_φ,

$$\mathbf{Coit}_\varphi(f)(a) = \mathrm{In}_\varphi(\mathbf{map}_\varphi(\mathbf{Coit}_\varphi(f))(f(a)))$$

Example: Signed digit representation

We are after a signed digit representation of real numbers x in the compact interval $\mathbb{I} := [-1, 1]$, i.e. we want

$$x = \sum_{n=0}^{\infty} d_n \cdot 2^{-(n+1)} \tag{1}$$

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(1) is equivalent to the fact that there are $x_0, x_1, \ldots \in \mathbb{I}$ such that $x = 1/2(d_0 + x_0) = 1/2(d_0 + 1/2(d_1 + x_1)) = \ldots$

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This suggests the following coinductive predicate on $\mathbb{I}$:

$$\mathrm{C}_0 = \nu X.\{x \mid \exists d \in \mathrm{SD}\, \exists x_0\, (x = \frac{d + x_0}{2} \wedge X(x_0)\}$$

The data type associated with C_0 is the type of infinite streams of signed digits. A stream $d_0, d_1, \ldots$ realizes $\mathrm{C}_0(x)$ precisely when (1) holds.

Extracting exact real number algorithms

Using coinduction one can prove, for example:

Theorem 1 $x \in \mathrm{C}_0$ iff $\forall n \in \mathbb{N}\, \exists q \in \mathbb{Q} \cap \mathbb{I}\, |x - q| \leq 2^{-n}$.

Theorem 2 If $x, y \in \mathrm{C}_0$ then $\frac{x+y}{2} \in \mathrm{C}_0$.

Theorem 3 If $x, y \in \mathrm{C}_0$ then $xy \in \mathrm{C}_0$.

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Similar implementations were studied by Edalat, Potts, Heckmann, Escardo, Marcial-Romero, Ciaffaglione, Gianantonio, . . .

The difference is that we *extract* the programs, together with their correctness proofs.

Characterizing uniform continuity by induction/coinduction

Recall the coinductive definition of reals in $\mathbb{I}$ that have a signed digit representation:

$$\mathrm{C}_0 = \nu X.\{x \mid \exists d \in \mathrm{SD}\, \exists x_0\, (x = \mathrm{av}_d(x_0) \wedge X(x_0)\}$$

where $\mathrm{av}_d(x_0) := \frac{d+x_0}{2}$.

We generalize this to a characterization of (uniformly) continuous functions $f : \mathbb{I} \to \mathbb{I}$:

$$\begin{aligned}\mathrm{C}_1 \quad = \quad \nu X.\mu Y.\{f \mid \exists d \in \mathrm{SD}\, \exists f_0\, (f = \mathrm{av}_d \circ f_0 \wedge X(f_0)) \\ \vee \ \forall d \in \mathrm{SD}\, Y(f \circ \mathrm{av}_d)\}\end{aligned}$$

The left disjunct is analogous to C_0 and means that f *emits* a digit.

The right disjunct means that f *absorbs* a digit from the input.

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Similar trees have been studied by P. Hancock, D. Pattinson, N. Ghani.

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The extracted program program has some similarity with A. Simpson's, but is more efficient because the functions to be integrated are represented differently.

Coalgebraic representation of functions

We represented several types of functions as coalgebras. The goal of this section is to move from these rather ad-hoc definitions to a more systematic way of defining such coalgebraic representation. The benefits will be:

- improved representations (regarding efficiency)
- natural extensions of these representations to functions of several arguments and higher type functionals,
- a new ultra-monomorphic and ultra-memoized model of functionals in higher types.

Representing functions on algebraic data by ν-types

The simplest example is the isomorphism

$$\mathbb{N} \to \beta \quad \simeq \quad \nu\alpha.\beta \times \alpha$$

i.e. the representation of function son the natural numbers as infinite streams.

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Altenkirch and Hinze showed that in general for any algebraic type A (built from $1, +, \times, \mu$) the functor $\lambda\beta.A \to \beta$ is naturally isomorphic to a coalgebraic functor (built from $\mathrm{id}, 1, +, \times, \nu$).

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Representing functions on algebraic data by ν-types

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- What if A is coalgebraic, i.e., contains ν?
- What if A itself is a function type?
- Is the stream representation really good? $\Longrightarrow$ demo

Functions vs streams

We make a comparison using the Fibonacci numbers modulo 2 and the constant function 0. Let $+_2$ denote addition modulo 2.

$$
\begin{aligned}
\text{fib } 0 &= 1 \\
\text{fib } 1 &= 1 \\
\text{fib } (n+2) &= \text{fib } n +_2 \text{fib } (n+1) \\
\text{const0 } n &= 0 \\
\text{fib}' &= 1 : 1 : \text{zipWith } (+_2)\ \text{fib}'\ (\text{tail fib}') \\
\text{const0}' &= 0 : \text{const0}'
\end{aligned}
$$

fib has exponential, fib' has linear time complexity.

const0 has constant, $\text{const0}'$ linear time complexity.

Can we find a natural representation of functions that performs optimally in both examples?

Towards a cartesian closed structure on $1+\times\mathrm{fix}$-types

Consider the domain T defined by the recursive domain equation

$$T = 1 + T \times T + T \times T$$

We regard the elements of T as (partial and not necessarily wellfounded) syntax for type expressions built from 1, $+$ and $\times$. Therefore, write the elements of T as $\bot$, $\mathbf{1}$, $\rho + \sigma$, $\rho \times \sigma$.

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Each $\rho \in T$ defines in a obvious way a subdomain $D(\rho) \subseteq D$ where D is defined by the recursive domain equation

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We will define a continuous function $\Rightarrow : [T] \to T \to T$ such that $\overline{D}(\vec{\rho} \Rightarrow \sigma)$ represents the space of sequential continuous functions from $\overline{D}(\vec{\rho})$ to $\overline{D}(\sigma)$.

Definition of $\Rightarrow : [T] \to T \to T$

The basic idea for the definition of $\vec{\rho} \Rightarrow \sigma$ is the same as for stream transformers: A function either *writes* (emits) a piece of output, or it *reads* (absorbs) a piece of input.

$$
\begin{aligned}
\_ \Rightarrow \mathbf{1} &= \mathbf{1} \\
[] \Rightarrow \sigma &= \sigma \\
\vec{\rho} \Rightarrow \sigma &= (\vec{\rho} \overset{\mathrm{W}}{\Rightarrow} \sigma) + (\vec{\rho} \overset{\mathrm{R}}{\Rightarrow} \sigma) \\
\vec{\rho} \overset{\mathrm{W}}{\Rightarrow} (\sigma_1 + \sigma_2) &\overset{!}{=} (\vec{\rho} \Rightarrow \sigma_1) + (\vec{\rho} \Rightarrow \sigma_2) \\
\vec{\rho} \overset{\mathrm{W}}{\Rightarrow} (\sigma_1 \times \sigma_2) &= (\vec{\rho} \Rightarrow \sigma_1) \times (\vec{\rho} \Rightarrow \sigma_2) \\
(\rho_1 + \rho_2), \vec{\rho} \overset{\mathrm{R}}{\Rightarrow} \sigma &= (\rho_1, \vec{\rho} \Rightarrow \sigma) \times (\rho_2, \vec{\rho} \Rightarrow \sigma) \\
(\rho_1 \times \rho_2), \vec{\rho} \overset{\mathrm{R}}{\Rightarrow} \sigma &= \rho_1, \rho_2, \vec{\rho} \Rightarrow \sigma
\end{aligned}
$$

Categorical combinators

To complete the ccc structure on has to define eval, composition, currying, etc. as suitable continuous functions on D indexed by elements of T.

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A partial prototype implementation exists. The implementation could be extracted from a proof that the $(D, \Rightarrow)$ defines a cartesian closed category. This is work in progress.

The implementation behaves indeed optimal in our examples ($\Longrightarrow$ demo).

Comparison of $N \to N$ in the stream model and the new model

Consider the unary natural numbers, $N \stackrel{\mu}{=} 1 + N$. According to Altenkirch/Hinze, the function space $N \to N$ is represented by infinite streams of natural numbers, $S \stackrel{\nu}{=} N \times S$.

The stream representation is eager, i.e. the input has to be read completely before any output is produced.

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The stream representation is eager, i.e. the input has to be read completely before any output is produced.

On the other hand, the representation $S := N \Rightarrow N$ yields

$$\begin{aligned} S &= W + R \\ W &\stackrel{\mu}{=} 1 + S \\ R &\stackrel{\nu}{=} N \times S \end{aligned}$$

In particular $S = 1 + S + N \times S$.

The extra components, $1+$ and $S+$ allow for an incremental construction of the output (possibly without reading the input). Hence, this representation is lazy.

Conclusion (program extraction)

Strengths

- Program extraction turns out to be very helpful (not a burden) in the example areas covered.
- New (correct!) programs have been extracted that would have been difficult to "guess".
- Using a fine tuning of realisability it is possible to do abstract mathematics as usual, and still get computational content.
- For example, there is no problem with using discontinuous and partial functions (sign function, least root of a polynomial).

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Open questions and further work

- Can we apply program extraction to areas that are less mathematical in nature?
- Can we address resource issues?
- Implementation of program extraction not yet complete.

Conclusion (mathematical spin-off)

The proof-as-programs paradigm is not only useful for program extraction, but also creates new mathematical ideas, methods and results.

For example:

- new methods and results in approximation- fixedpoint- and ergodic-theory
- Memoized computation in higher types
- New forms of bar recursion
- Selection functions
- New "computationally efficient" definitions of uniform continuity
- Uniform logical connectives

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