# Density theorems for the domains-with-totality semantics of dependent types

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**Abstract.** We study a semantics of dependent types and universe operators based on parametrized domains with totality. The main results are generalizations of the Kleene/Kreisel density theorem for the continuous functionals. This continues work of E. Palmgren and V. Stoltenberg–Hansen on the domain interpretation of dependent types, and of D. Normann on universes of wellfounded types with density.

**Key words:** Continuous functionals, Domains, Totality, Dependent types, Universes

## 1. Introduction

In Mathematical Logic and Computer Science there is growing interest in constructive type theories as developed by Martin–Löf [8]. This paper is concerned with a semantics of such theories within the realm of Ershov–Scott domains [5] with totality [10].

Erik Palmgren and Viggo Stoltenberg–Hansen [15], [17] developed a semantics for a *partial* type theory (modelling partial functions and functionals) based on the notion of a *parametrization*, i.e. a domain depending on parameters. Since this semantics was very natural and elegant it was natural to ask whether this could be modified in order to get a semantics for *total* type theory (modelling total functionals or type theory as a logical system) The obvious choice was to interpret a type by a domain $D$ together with a subset $D_{\text{tot}} \subseteq D$ of "total" objects, e.g. $D = \mathbb{N}_\perp$ and $D_{\text{tot}} = \mathbb{N}$, and modelling a dependent type by a parametrization $F\colon D \to \text{DOM}$ (DOM = the category of domains with embeddings) together with a subset $F_{\text{tot}}(x) \subseteq F(x)$ for each total $x \in D_{\text{tot}}$. Dag Norman together with Lill Kristiansen and Geir Waagbø developed such a semantics in a series of papers, e.g. [10], [11], [20].

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The constructions in [10], [11], [20] which we are going to analyze may be described roughly as follows. We define inductively a domain $S$ of *syntactic forms* and for every $s \in S$ a domain $I(s)$, the *interpretation* of $s$, such that $\{I(s) : s \in S\}$ will form a universe of types closed under dependent products and sums. To start with we let $\bot \in S$ with $I(\bot) := \{\bot\}$, and $\mathrm{nat}, \mathrm{boole} \in S$ with

$$\begin{aligned} I(\mathrm{nat}) &:= \mathbb{N}_\bot = \{\bot\} \cup \mathbb{N}, \\ I(\mathrm{boole}) &:= \mathbb{B}_\bot = \{\bot, \#\mathrm{t}, \#\mathrm{f}\}. \end{aligned}$$

Inductively, if we have $s \in S$ and a continuous function $f: I(s) \to S$, then we let $(\pi, s, f), (\sigma, s, f) \in S$ and

$$\begin{aligned} I(\pi, s, f) &:= (\Pi x \in I(s)) I(f(s)), \\ I(\sigma, s, f) &:= (\Sigma x \in I(s)) I(f(s)). \end{aligned}$$

This defines the partial universe $(S, I)$. Now we define inductively the set $S_{\mathrm{wf}} \subseteq S$ of *wellfounded types* and for each $s \in S_{\mathrm{wf}}$ the set $I_{\mathrm{tot}}(s) \subseteq I(s)$ of *total elements*. We let $\mathrm{nat}, \mathrm{boole} \in S_{\mathrm{wf}}$ and

$$\begin{aligned} I_{\mathrm{tot}}(\mathrm{nat}) &:= \mathbb{N}, \\ I_{\mathrm{tot}}(\mathrm{boole}) &:= \mathbb{B} := \{\#\mathrm{t}, \#\mathrm{f}\}. \end{aligned}$$

Inductively, if $s \in S_{\mathrm{wf}}$ and $f: I(s) \to S$ is a continuous function such that $f(x) \in S_{\mathrm{wf}}$ for all $x \in I_{\mathrm{tot}}(s)$, then we let $(\pi, s, f)$,$(\sigma, s, f) \in S_{\mathrm{wf}}$ and

$$\begin{aligned} I_{\mathrm{tot}}(\pi, s, f) &:= \{z \in I(\pi, s, f) : \forall x \in I_{\mathrm{tot}}(s). z(x) \in I_{\mathrm{tot}}(f(s))\}, \\ I_{\mathrm{tot}}(\sigma, s, f) &:= \{(x, u) \in I(\sigma, s, f) : x \in I_{\mathrm{tot}}(s) \wedge u \in I_{\mathrm{tot}}(f(s))\}. \end{aligned}$$

In [10], [11] Normann proved that $S_{\mathrm{wf}}$ is a dense subset of $S$. Order theoretically this means that to every finite (compact) object $s_0 \in S$ there is a wellfounded $s \in S_{\mathrm{wf}}$ such that $s_0 \sqsubseteq s$. Furthermore for every $s \in S_{\mathrm{wf}}$ the set of total objects $I_{\mathrm{tot}}(s)$ is a dense subset of $I(s)$.

We are interested in density since it is the key to many of the deeper results on the total semantics. We mention the most important:

- *Choice.* If an effectively continuous function $f: D \times E \to \mathbb{N}_\bot$ is defined for all total arguments and $\forall x \in D_{\mathrm{tot}} \exists y \in E_{\mathrm{tot}}(f(x, y) = 0)$, then there is an effectively continuous choice function $g: D \to E$ such that $\forall x \in D_{\mathrm{tot}}(f(x, g(x)) = 0)$. This was used by Kreisel in [7].
- *Extensionality.* The order theoretic consistency relation between total objects is an equivalence relation and coincides with extensional equality.

- *Realizability.* A suitable realizability semantics [2], [19] and the total domain semantics restricted to hereditarily computable objects coincide. For simple types this has been proven by Ershov [4] and in an abstract form in [1] as a generalisation of the Kreisel–Lacombe–Shoenfield theorem. For dependent types this is still open.
- *Hierarchies.* Using density we can compare the complexity of the total semantics with hierarchies in generalized recursion theory [14].

The density theorem for simple, i.e. non–dependent types goes back to Kleene [6] and Kreisel [7] and has been put into a domain theoretic context by Ershov [3]. In this paper we will prove a generalization of Normann's density theorem for dependent types implying density theorems for universes closed under $\Pi$, $\Sigma$ and further operators like the W–type or universe operators.

Our result can be described roughly as follows. If instead of $\mathbb{N}$ and $\mathbb{B}$ we start our hierarchy of types with some arbitrary family $B = (B(a))_{a\in A}$ of base types, we get a universe $I = (I_B(s))_{s\in S_B}$ which depends on $B$. The mapping sending $B$ to $I_B$ defines a continuous functor, call it $\mathcal{U}_1$, the 'first universe operator'. Inductively, the $n+1$st universe operator $\mathcal{U}_{n+1}$ takes a family of base types $B$ and closes it under $\Pi$, $\Sigma$ and the previously defined universe operators $\mathcal{U}_1, \ldots, \mathcal{U}_n$. Let $(I_n(s))_{s\in S_n} = \mathcal{U}_n(\mathbb{N}, \mathbb{B})$. We prove:

> For every $n$, the wellfounded objects in $S_n$ are dense and co–dense and for every wellfounded $s \in S_n$ the total objects in $I_n(s)$ are dense and co–dense.

The result could be extended easily to transfinite iterations of universe operators. It also holds if in addition we close under (a modification of) the W–type.

In [1] we proved a density theorem for the function space, i.e. the non-dependent product. The main difficulty in moving from the non-dependent to the dependent case is to express in the right way what it means that the set of total objects $I_{\mathrm{tot}}(s)$ is dense in the domain $I(s)$ *uniformly in s*. The solution to this problem can be described roughly as follows. Let $D$ a domain, $D_{\mathrm{tot}} \subseteq D$, $F\colon D \to \mathrm{DOM}$ a parametrization and $F_{\mathrm{tot}}(x) \subseteq F(x)$ for each $x \in D_{\mathrm{tot}}$. We assume further that we have a continuous function $K\colon \Sigma(D, F) \to \Pi(D, F)$ with certain additional properties. We call $K$ a *transporter* since it transports an element $u \in F(x)$ to an element $K(x, u)(y) \in F(y)$. Note that $F$, since it is a functor, transports $u \in F(x)$ to $F[x, y](u) \in F(y)$ but only if $x \sqsubseteq y$. The properties of $K$ will ensure that in this case $K(x,u)(y) = F[x,y](u)$. We call the family $(F_{\text{tot}}(x))_{x\in D_{\text{tot}}}$ *uniformly dense* in $F$ if for each finite $x_0 \in D$ and each finite $u_0 \in F(x_0)$ there is some continuous choice function $d \in \Pi(D,F)$ such that

$$\forall x \in D(K(x_0,u_0)(x) \sqsubseteq d(x)) \quad \text{and} \quad \forall x \in D_{\text{tot}}(d(x) \in F_{\text{tot}}(x)).$$

Since every finite element $v_0 \in F(x)$ is of the form $v_0 = K(x_0,u_0)(x)$ for some finite $x_0 \sqsubseteq x$ and finite $u_0 \in F(x_0)$ we get for total $x$

$$v_0 = K(x_0,u_0)(x) \sqsubseteq d(x) \in F_{\text{tot}}(x).$$

Hence uniform density implies density of $F_{\text{tot}}(x)$ in $F(x)$ pointwise. A similar "uniformization" is possible for the associated notion of *co–density* (called "totality" in[1]).

We call an operator perfect if it, roughly speaking, preserves uniform density and co–density. The *density theorem for* $\Pi$ *and* $\Sigma$ (theorem 2) states that $\Pi$ and $\Sigma$ are perfect operators. In the *density theorem for universes* (theorem 3) we show that a universe operator performing closure under perfect operators is itself perfect.

This leads to complex hierarchies of domains with dense totalities. In [14] Normann proved that the hierarchy generated by $\Pi$ and $\Sigma$ has the same closure ordinal as Kleene Recursion in ${}^3E$. For stronger systems the relations to recursion theoretic hierarchies are still unknown. We hope that the results in this paper will bring us closer to a solution of these problems.

## 2. Dependent domains

We will mainly use notations and results from [5] and [15] concerning the basics of domains and dependent domains, i.e. parametrizations.

By $D$ and $E$ we denote arbitrary Ershov-Scott domains. $[D \to E] := \{f\colon D \to E : f \text{ continuous}\}$, $D \times E := \{(x,y) : x \in D, y \in E\}$, $D + E := \{0\} \times D \cup \{1\} \times E \cup \{\bot\}$ with the usual orderings. In $D+E$ we will sometimes use more suggestive labels than 0 and 1. $D_0$ denotes the set of compacts of $D$. DOM denotes the category of domains with embeddings as morphisms. If $\eta\colon D \to E$ is an embedding then $\eta^-\colon E \to D$ denotes the associated projection with $\eta^- \circ \eta = \text{id}_D$ and $\eta \circ \eta^- \sqsubseteq \text{id}_E$. $D \times E$ is the categorical product of $D$ and $E$, whereas $D + E$ is *not* the categorical coproduct of $D$ and $E$; in fact DOM has no coproducts. Also DOM is not cartesian closed. However DOM has direct colimits.

A *parametrization* $(D, F)$ consists of a domain $D$ and a continuous functor $F\colon D \to \mathrm{DOM}$, where $D$ is considered as a category in the usual way. See [15] for useful characterizations of parametrizations. Sometimes we will write just $F$ instead of $(D, F)$. If $x \sqsubseteq y \in D$ and $u \in F(x)$ then $u^{(y)} := F[x, y](u) \in F(y)$, where $[x, y]$ is the unique morphism from $x$ to $y$, and for $v \in F(y)$, $v_{(x)} := F[x, y]^{-}(v) \in F(x)$.

Simple examples of parametrizations are obtained as follows. Let $D^1, \dots, D^k$ be domains. Define the parametrization

$$\langle D^1, \dots, D^k \rangle := (\{1, \dots, k\}_\perp, F),$$

where $F(\perp) := \{\perp\}$ and $F(i) := D^i$.

The domains $\Pi(D, F)$ (dependent product) and $\Sigma(D, F)$ (dependent sum) are defined as in [15]:

$$\Pi(D, F) = \{f \in \Pi_{x \in D} F(x) : f \text{ monotone and continuous}\},$$

$$f \sqsubseteq g \Leftrightarrow \forall x \in D. f(x) \sqsubseteq g(x).$$

$$\Sigma(D, F) = \{(x, u) : x \in D, u \in F(x)\},$$

$$(x, u) \sqsubseteq (y, v) \Leftrightarrow x \sqsubseteq y \text{ and } u^{(y)} \sqsubseteq v.$$

Here, '$f$ monotone' means $\forall x, y \in D. x \sqsubseteq y \Rightarrow f(x)^{(y)} \sqsubseteq f(y)$ and '$f$ continuous' means $f(\bigsqcup A) = \bigsqcup\{f(x)^{(\bigsqcup A)} : x \in A\}$ for each directed set $A \subseteq D$. Instead of $\Pi(D, F)$ we will sometimes write $(\Pi x \in D)F(x)$ if this improves readability.

If $F\colon D \to \mathrm{DOM}$ and $G\colon \Sigma(D, F) \to \mathrm{DOM}$ are parametrizations then the parametrizations $\Pi(F, G)\colon D \to \mathrm{DOM}$ and $\Sigma(F, G)\colon D \to \mathrm{DOM}$ are defined by

$$\begin{aligned} \Pi(F, G)(x) &= \Pi(F(x), \lambda u. G(x, u)), \\ \Pi(F, G)[x, y](f) &= \lambda v \,.\, f(v_{(x)})^{(y, v)}, \end{aligned}$$

$$\begin{aligned} \Sigma(F, G)(x) &= \Sigma(F(x), \lambda u. G(x, u)), \\ \Sigma(F, G)[x, y](u, r) &= (u^{(y)}, r^{(y, u^{(y)})}). \end{aligned}$$

*Definition 1.* The category PAR has parametrizations $(D, F)$ for objects, and morphisms $(\eta, \tau)\colon (D, F) \to (E, G)$ where $\eta\colon D \to E$ is an embedding (i.e. a morphism in DOM) and $\tau\colon F \to G \circ \eta$ is a natural transformation. Composition of morphisms is defined by

$$(\eta_1, \tau_1) \circ (\eta, \tau) = (\eta_1 \circ \eta, \lambda x. \tau_1(\eta(x)) \circ \tau(x)).$$

*Remark.* The alternative choice suggested by Palmgren to take morphisms $(\eta, \sigma)\colon (D, F) \to (E, G)$ with a natural transformation $\sigma\colon (E, F \circ \eta^-) \to (E, G)$ leads to an isomorphic category.

*Lemma 1.* (a) Let $(D, F), (D, G) \in \mathrm{PAR}$ and let $\tau\colon F \to G$ be a natural transformation. Then $\tau$ is continuous, i.e.

$$\tau \in (\Pi x \in D)[F(x) \to G(x)].$$

(b) Let $(\eta, \tau)\colon (D, F) \to (E, G)$ be a PAR–morphism. Then the functions $\Pi(\eta, \tau)\colon \Pi(D, F) \to \Pi(E, G)$ and $\Sigma(\eta, \tau)\colon \Sigma(D, F) \to \Sigma(E, G)$ defined by

$$\begin{aligned} \Pi(\eta, \tau)(f) &= \lambda y \in E \,.\, \tau(\eta^-(y))(f(\eta^-(y)))^{(y)}, \\ \Sigma(\eta, \tau)(x, u) &= (\eta(x), \tau(x)(u)) \end{aligned}$$

are embeddings.

*Proof.* For (a) a characterization of continuity for functors $F\colon D \to \mathrm{DOM}$ in [15] is used. (b) follows easily from (a).

*Definition 2.* We define the following continuous functors.

(i) $\mathrm{dom}\colon \mathrm{PAR} \to \mathrm{DOM}$, $\mathrm{dom}(D, F) = D$, $\mathrm{dom}(\eta, \tau) = \eta$.

(ii) $\Pi\colon \mathrm{PAR} \to \mathrm{DOM}$ and $\Sigma\colon \mathrm{PAR} \to \mathrm{DOM}$. $\Pi(\eta, \tau)$ and $\Sigma(\eta, \tau)$ are defined in lemma 1.

(iii) $+\colon \mathrm{PAR}^n \to \mathrm{PAR}$, $(D_1, F_1) + \ldots + (D_n, F_n) := (D_1 + \ldots + D_n, G)$ where $G(i, x_i) := F_i(x_i)$ and $G(\bot) := \{\bot\}$. On morphisms $+$ is defined in the obvious way.

*Definition 3.* Let $\Phi\colon \mathrm{PAR} \to \mathrm{DOM}$ be a continuous functor. We define a continuous functor $\Phi!\colon \mathrm{PAR} \to \mathrm{PAR}$ as follows.

$$\mathrm{dom}(\Phi!(D, F)) = (\Sigma x \in D)[F(x) \to D],$$

$$\Phi!(D, F)(x, f) = \Phi(F(x), F \circ f),$$

$$\Phi!(D, F)[(x, f), (y, g)] = \Phi(F[x, y], \lambda u.F[f(u), g(u^{(y)})]),$$

and if $(\eta, \tau)\colon (D, F) \to (E, G)$ is a morphism then $\Phi!(\eta, \tau) = (\eta_1, \tau_1)$, where $(\eta_1, \tau_1)\colon \Phi!(D, F) \to \Phi!(E, G)$ is defined by

$$\eta_1(x, f) = (\eta(x), \eta \circ f \circ \tau^-(x)), \quad \tau_1(x, f) = \Phi(\tau(x), \tau \circ f).$$

## 3. Universes

We will construct universes and universe operators as least solutions to 'parametrization equations', i.e. least fixed-points of continuous functors.

*Theorem 1.* (1) Every continuous functor $\Psi\colon \mathrm{PAR} \to \mathrm{PAR}$ has an initial fixed point, denoted $\mathrm{fix}\, F.\Psi(F) \in \mathrm{PAR}$.

(2) To every continuous functor $\Psi\colon \mathrm{PAR} \times \mathrm{PAR} \to \mathrm{PAR}$ there is a continuous functor $\mathcal{U}\colon \mathrm{PAR} \to \mathrm{PAR}$ such that for all $B \in \mathrm{PAR}$

$$\mathcal{U}(B) = \mathrm{fix}\, F.\Psi(B, F).$$

More precisely there is a natural isomorphism between the functors $\mathcal{U}$ and $\lambda B.\Psi(B, \mathcal{U}(B))$.

*Proof.* It is easy to see that, like DOM, the category PAR has direct colimits. It's a folklore result in category theory that then every continuous functor has an initial fixed point which depends continuously on parameters. □

*Definition 4.* Let $(A, B) \in \mathrm{PAR}$ and let $\vec{\Phi} = \Phi^1, \ldots, \Phi^k$ continuous functors from PAR to DOM. The *universe over* $(A, B)$ *closed under* $\vec{\Phi}$, is defined by

$$\mathcal{U}[\vec{\Phi}](A, B) := \mathrm{fix}\,(S, I)\,.\,(A, B) + \Phi^1!(S, I) + \ldots + \Phi^k!(S, I).$$

By theorem 1(1) this is well defined, and by theorem 1(2) this defines a continuous functor $\mathcal{U}[\vec{\Phi}]\colon \mathrm{PAR} \to \mathrm{PAR}$.

To explain the construction in more detail we let

$$(S, I) := \mathcal{U}[\vec{\Phi}](A, B).$$

Following [10] and [11] we call $S \in \mathrm{DOM}$ the domain of *syntactic forms* or codes of types and $I\colon S \to \mathrm{DOM}$ the *interpretation map.* We have

$$S \simeq +A + (\Sigma s \in S)[I(s) \to S] + \ldots + (\Sigma s \in S)[I(s) \to S],$$

where "$\simeq$" denotes isomorphism of domains. We denote the isomorphism from left to right by $\tau$. Hence for every $s \in S$, $\tau(s)$ is of one of the following forms (using suggestive labels $\beta, \varphi_1, \ldots \varphi_k$): $\perp$, or $(\beta, a)$ where $a \in A$, or $(\varphi_i, s_1, f)$ where $s_1 \in S$ and $f \in [I(s_1) \to S]$.

The interpretation map $I\colon S \to \mathrm{DOM}$ satisfies

$$\text{if } \tau(s) = \bot \text{ then } I(s) \simeq \{\bot\}.$$

$$\text{if } \tau(s) = (\beta, a) \text{ then } I(s) \simeq B(a),$$

$$\text{if } \tau(s) = (\varphi_i, s_1, f) \text{ then } I(s) \simeq \Phi^i(I(s_1), I \circ f).$$

The inverse of $\tau$ is given by injective continuous functions $\beta\colon A \to S$ and $\varphi_i\colon (\Sigma s \in S)[I(s) \to S] \to S$ (note the overloading with the labels) such that

$$\tau(\beta(a)) = (\beta, a) \text{ for all } a \in A,$$

$$\tau(\varphi_i(s, f)) = (\varphi_i, s, f) \text{ for all } s \in S \text{ and } f \in [I(s) \to S].$$

Since $(S, I)$ is the least fixed point of a continuous functor, this fixed point is reached after $\omega$ iterations. Hence to every compact $s_0 \in S_0$ we may assign a rank $\mathrm{rk}(s_0) \in \mathbb{N}$, the stage when $s_0$ comes in first, such that if $\tau(s_0) = (\varphi_i, s_1, f)$ then $\mathrm{rk}(s_1) < \mathrm{rk}(s_0)$ and, for all $x \in I(s_1)$ the syntactic form $f(x)$ is compact with $\mathrm{rk}(f(x)) < \mathrm{rk}(s_0)$.

*Remark.* In [10], [11] etc. a concrete construction of $(S, I)$ for $\vec{\Phi} = \Pi, \Sigma$ is given. "Concrete" means that the elements of $S_0$ and $I(s_0)_0$ and their order relation are constructed directly by an inductive definition. In [17] similar constructions are studied in the framework of information system which are concrete representations of domains.

*Definition 5.* (i) The *universes* $(S^{(n)}, I^{(n)}) \in \mathrm{PAR}$ are defined by

$$(S^{(n)}, I^{(n)}) := \mathcal{U}[\Pi, \Sigma]\langle \mathbb{N}_\bot, \mathbb{B}_\bot, S^{(1)}, \ldots, S^{(n-1)}\rangle.$$

This corresponds to Martin–Löf theories with $n$ universes (see e.g. [16], [18]).

(ii) The *iterated universe operators* $\mathcal{U}^n\colon \mathrm{PAR} \to \mathrm{PAR}$ are defined by

$$\mathcal{U}^n := \mathcal{U}[\Pi, \Sigma, \mathrm{dom} \circ \mathcal{U}^1, \ldots, \mathrm{dom} \circ \mathcal{U}^{n-1}].$$

We let $(S^n, I^n) := \mathcal{U}^n\langle \mathbb{N}^\bot, \mathbb{B}^\bot\rangle$ and call it the $n$th *super universe.*

One might wonder whether these universes have indeed 'enough' closure properties, since e.g. we put only a code of $S^{(k)}$ $(k < n)$ into $S^{(n)}$, but *not* codes of $I^{(k)}(s)$ for $s \in S^{(k)}$. Similarly we closed $(S^n, I^n)$ only under $\mathrm{dom} \circ \mathcal{U}^k$ $(k < n)$ but *not* under $\mathcal{U}^k$. The next lemma says that the desired closure properties do nevertheless hold.

*Lemma 2.* Let $k < n \in \mathbb{N}$. There are $i^{(k)} \in [S^{(k)} \to S^{(n)}]$ and $j^{(k)} \in [S^{(k)} \to S^2]$ such that

$$I^{(k)}(s) = I^{(n)}(i^{(k)}(s)) = I^2(j^{(k)}(s)$$

for all $s \in S^{(k)}$. Furthermore there is a continuous function $u^k \in [(\Sigma s \in S^n)(\Sigma f \in [I^n(s) \to S^n])(\Sigma t \in \mathrm{dom}\mathcal{U}^k(I^n(s), I^n \circ f)) \to S^n]$ such that

$$\mathcal{U}^k(I^n(s), I^n \circ f)(t) = I^n(u^k(s, f, t))$$

for all legal arguments $s, f, t$.

*Proof.* We will not use this lemma in the sequel. Hence we confine ourselves with a sketch of the definition of $i^{(k)}$ ($j^{(k)}$ and $u^k$ can be defined similarly). First we define in an obvious way $i^{(k)}(s_0)$ for compact $s_0 \in S_0^{(k)}$ only, such that $I^{(k)}(s_0) = I^{(n)}(i^{(k)}(s_0))$. The definition proceeds by recursion on the rank of $s_0$. It's clear that $i^{(k)}$ is monotone on compacts. Hence we may extend $i^{(k)}$ to *all* $s \in S^{(k)}$ in the usual way. By continuity of $I^{(k)}$ and $I^{(n)}$ the desired equation $I^{(k)}(s) = I^{(n)}(i^{(k)}(s))$ holds for all $s \in S^{(k)}$. □

## 4. Totality

*Definition 6.* (i) A *category with totality* consists of a category $\mathcal{C}$ together with an assignment of a set $\mathcal{C}_*(a)$ to every object $a \in \mathcal{C}$. If $a_* \in \mathcal{C}_*(a)$ we say "$a_*$ is a totality on $a$".

(ii) A *totality on a domain D* is a subset of $D$, i.e. we obtain a category with totality by letting $\mathrm{DOM}_*(D) :=$ the power set of $D$.

(iii) A *totality on a parametrization* $(D, F) \in \mathrm{PAR}$ is a pair $(L, M)$ such that $L \subseteq D$ and $M = (M(x))_{x \in L}$ is a family such that $M(x) \subseteq F(x)$ for each $x \in L$. Instead of "$(L, M)$ is a totality on $(D, F)$" we also write "$M \subseteq_L F$". This defines a category with totality $(\mathrm{PAR}, \mathrm{PAR}_*)$.

(iv)If $\mathcal{C}, \mathcal{D}$ are categories with totality, then $\Psi_*$ is a *totality on a functor* $\Psi: \mathcal{C} \to \mathcal{D}$ if $\Psi_*$ maps every totality on an object $a \in \mathcal{C}$ to a totality on $\Psi(a)$. This extends in an obvious way to functors with more than one argument.

*Definition 7.* The standard totalities on the domains $\mathbb{N}_\bot = \mathbb{N} \cup \{\bot\}$ and $\mathbb{B}_\bot = \{\bot, \#\mathrm{t}, \#\mathrm{f}\}$ are $\mathbb{N}$ and $\mathbb{B} = \{\#\mathrm{t}, \#\mathrm{f}\}$ respectively.

If $D_*^1 \subseteq D^1, \ldots, D_*^k \subseteq D^k$ are totalities on domains then we define on the parametrization $\langle D^1, \ldots, D^k \rangle$ the totality

$$\langle D_*^1, \ldots, D_*^k \rangle := (\{1, \ldots, k\}, F_*)$$

by $F_*(i) := D^i_*$.

For all continuous functors $\Phi$ defined in the previous section we have standard totalities $\Phi_*$. The standard totalities on $\Pi, \Sigma, \mathrm{dom}\colon \mathrm{PAR} \to \mathrm{DOM}$ are given given as follows. Let $(L, M)$ be a totality on $(D, F)$.

$$\begin{aligned}
\Pi_*(L,M) &:= \{f \in \Pi(D,F) : \forall x \in L \,.\, f(x) \in M(x)\},\\
\Sigma_*(L,M) &:= \{(x,u) \in \Sigma(D,F) : x \in L \wedge u \in M(x)\},\\
\mathrm{dom}_*(L,M) &:= L.
\end{aligned}$$

Considering the functors $\to, \times, +$ on domains as special cases of $\Pi$ and $\Sigma$ we see that their standard totalities map $L \subseteq D$ and $M \subseteq E$ to

$$\begin{aligned}
[L \to_* M] &= \{f \in [D \to E] : f[L] \subseteq M\},\\
L \times_* M &= \{(x,y) : x \in L, y \in M\},\\
L +_* M &= \{0\} \times L \cup \{1\} \times M.
\end{aligned}$$

Finally the standard totality on $+\colon \mathrm{PAR}^n \to \mathrm{PAR}$ is defined by $(L_1, M_1) +_* \ldots +_* (L_n, M_n) := (L_1 +_* \ldots +_* L_n, M)$, where $M(i, x_i) := M_i(x_i)$.

*Definition 8.* Let $\Phi\colon \mathrm{PAR} \to \mathrm{DOM}$ be a continuous functor and $\Phi_*$ a totality on $\Phi$. We define a totality $\Phi_*!$ on $\Phi!$ as follows. Let $(D, F) \in \mathrm{PAR}$ and let $(L, M)$ be a totality on $(D, F)$. Then

$$\Phi_*!(L,M) := ((\Sigma_* x \in L)[M(x) \to_* L], \lambda(x,f).\Phi_*(M(x), M \circ f)).$$

Let $B\colon A \to \mathrm{DOM}$ be a parametrization and $\vec{\Phi} = \Phi^1, \ldots, \Phi^k\colon \mathrm{PAR} \to \mathrm{DOM}$ continuous functors. Consider the universe $(S, I) = \mathcal{U}[\vec{\Phi}](B)$. Given totalities $(A_*, B_*)$ on $(A, B)$ and $\Phi^i_*$ on $\Phi^i$ we will define by a least–fixed–point construction a totality $(S_{\mathrm{wf}}, I_{\mathrm{tot}})$ on $(S, I)$ corresponding to Normann's wellfounded types [10],[11].

*Definition 9.* We define orderings on totalities

(i) The ordering on totalities on a domain is just set inclusion.

(ii) Totalities on a parametrization are ordered by graph inclusion, i.e. $(L_1, M_1) \subseteq (L_2, M_2)$ iff $L_1 \subseteq L_2$ and $M_1(x) = M_2(x)$ for all $x \in L_1$.

(iii) Totalities on a functor $\Psi\colon \mathcal{C} \to \mathcal{D}$ are ordered pointwise, i.e. $\Psi^1_* \subseteq \Psi^2_*$ iff $\Psi^1_*(x_*) \subseteq \Psi^2_*(x_*)$ for all totalities $x_*$ on an object $x \in \mathcal{C}$.

A totality $\Psi_*$ on $\Psi$ is *monotone* if $x^1_* \subseteq x^2_*$ implies $\Psi_*(x^1_*) \subseteq \Psi_*(x^2_*)$ for all totalities $x^1_*, x^2_*$ on an object $x \in \mathcal{C}$.

*Lemma 3.* (a) The totalities $\mathrm{dom}_*$, $\Sigma_*$ and $+_*$ are monotone.

(b) If $\Phi_*$ is a totality on a functor $\Phi: \mathrm{PAR} \to \mathrm{DOM}$ or $\Phi: \mathrm{PAR} \to \mathrm{PAR}$ then $\Phi_*!$ is monotone.

*Proof.* Easy.

Note that the standard totality on $\Pi$ is not monotone.

*Lemma 4.* Let $\mathcal{C}$ be one of the categories with totality considered so far. Let $\Psi: \mathcal{C} \to \mathcal{C}$ be a functor and $x \in \mathcal{C}$ a fixed point of $\Psi$. Let $\Psi_*$ be a monotone totality on $\Psi$. Then there is a least totality $x_*$ on $x$ such that $\Psi_*(x_*) = x_*$.

*Proof.* Clearly the totalities on an object $x \in \mathcal{C}$ form a cpo with least element. Hence we can apply the Knaster–Tarski fixed point theorem.

*Definition 10.* Let $(S, I) = \mathcal{U}[\vec{\Phi}, \vec{\Psi}](A, B)$, i.e. $(S, I) \simeq \Psi(S, I)$, where the continuous functor $\Psi: \mathrm{PAR} \to \mathrm{PAR}$ is defined by

$$\Psi(D, F) = (A, B) + \Phi^1!(D, F) + \ldots + \Phi^k!(D, F).$$

Let $(A_*, B_*)$ be a totality on $(D, F)$ and $\Phi^i_*$ a totality on $\Phi^i$. We define a totality $\Psi_*$ on $\Psi$ by

$$\Psi_*(L, M) = (A_*, B_*) +_* \Phi^1_*!(L, M) +_* \ldots +_* \Phi^k_*!(L, M).$$

(i) A totality on $(S, I)$ is *admissible* if it is a fixed point of $\Psi_*$.

(ii) By lemma 3 $\Psi_*$ is monotone. We let $(S_{\mathrm{wf}}, I_{\mathrm{tot}})$ be the least fixed point of $\Psi_*$ which exists by lemma 4. $(S_{\mathrm{wf}}, I_{\mathrm{tot}})$ is the least admissible totality on $(S, I)$. We call it the *wellfounded totality.*

(iii) The *wellfounded totality* on the continuous functor $\mathcal{U}[\vec{\Phi}]: \mathrm{PAR} \to \mathrm{PAR}$ is defined by

$$\mathcal{U}_{\mathrm{wf}}[\vec{\Phi}_*](A_*, B_*) := (S_{\mathrm{wf}}, I_{\mathrm{tot}}).$$

(iv) The *wellfounded iterated universes operators* $\mathcal{U}^{(n)}_{\mathrm{wf}}$ are defined by

$$\mathcal{U}^n_{\mathrm{wf}} := \mathcal{U}_{\mathrm{wf}}[\Pi_*, \Sigma_*, \mathrm{dom}_* \circ \mathcal{U}^1_{\mathrm{wf}}, \ldots, \mathrm{dom}_* \circ \mathcal{U}^{n-1}_{\mathrm{wf}}].$$

(v) The *wellfounded universes* $(S^{(n)}_{\mathrm{wf}}, I^{(n)}_{\mathrm{tot}})$ and the *wellfounded super universes* $(S^n_{\mathrm{wf}}, I^n_{\mathrm{tot}})$ are defined by

$$\begin{aligned}(S^{(n)}_{\mathrm{wf}}, I^{(n)}_{\mathrm{tot}}) &:= \mathcal{U}_{\mathrm{wf}}[\Pi_*, \Sigma_*]\langle \mathbb{N}, \mathbb{B}, S^{(1)}_{\mathrm{wf}}, \ldots, S^{(n-1)}_{\mathrm{wf}} \rangle, \\ (S^n_{\mathrm{wf}}, I^n_{\mathrm{tot}}) &:= \mathcal{U}^n_{\mathrm{wf}}\langle \mathbb{N}, \mathbb{B} \rangle.\end{aligned}$$

Being a little bit sloppy we may define $S_{\mathrm{wf}} \subseteq S$ (i (iii) above) inductively together with $I_{\mathrm{tot}}(s) \subseteq I(s)$ for each $s \in S_{\mathrm{wf}}$ as follows (compare with the introduction).

If $\tau(s) = (\beta, a)$, where $a \in A_*$ then $s \in S_{\mathrm{wf}}$ and $I_{\mathrm{tot}}(s) \simeq B_*(a)$.

If $\tau(s) = (\varphi_i, s_1, f)$, where $s_1 \in S_{\mathrm{wf}}$ and $f \in [I_{\mathrm{tot}}(s_1) \to_* S_{\mathrm{wf}}]$ then $s \in S_{\mathrm{wf}}$ and

$$I_{\mathrm{tot}}(s) \simeq \Phi^i_*(I_{\mathrm{tot}}(s_1), \lambda x.I_{\mathrm{tot}}(f(x))).$$

A totality $(S_*, I_*)$ is admissible iff the following hold.

$\bot \notin S_*$.

If $\tau(s) = (\beta, a)$ then $s \in S_*$ iff $a \in A_*$ and in that case $I_*(s) \simeq B_*(a)$.

If $\tau(s) = (\varphi_i, s_1, f)$ then $s \in S_*$ iff $s_1 \in S_*$ and $f \in [I_*(s_1) \to_* S_*]$ and in that case $I_*(s) \simeq \Phi^i_i(I_*(s_1), \lambda x.I_*(f(x)))$.

In general no admissible totalities different from $(S_{\mathrm{wf}}, I_{\mathrm{tot}})$ must exist. However if all $\Phi^i_*$ are monotone w.r.t. the ordering

$$(L_1, M_1) \leq (L_2, M_2) :\Leftrightarrow L_1 = L_2 \wedge \forall x \in L_1.M_1(x) \subseteq M_2(x)$$

on totalities on a parametrization, then we may define a larger admissible totality corresponding to Normann's type streams [9], [13]. For instance the standard totalities on $\Pi$ and $\Sigma$ are monotone w.r.t. $\leq$.

## 5. Transporters

In order to define a suitable notion of uniform density we need an extra structure on parametrizations $F\colon D \to \mathrm{DOM}$. By the functoriality of $F$, if $x \sqsubseteq y \in D$ then $F(x)$ and $F(y)$ are connected via the embedding $F[x, y]\colon F(x) \to F(y)$. It turns out that we need a connection also in case when $x$ and $y$ are not related. We need to be able to transport $u \in F(x)$ into any $F(y)$ we like.

*Definition 11.* For a parametrization $(D, F) \in \mathrm{PAR}$ we let

$$\mathcal{K}(D, F) := [\Sigma(D, F) \to \Pi(D, F)].$$

Let $K \in \mathcal{K}(D, F)$ and $x, y \in D$. $K$ is an *$x, y$–transporter* if for all $u \in F(x)$ and $z \in D$

(i) if $x = y$ then $K(x,u)(y) = u$,

(ii) if $x \sqsubseteq z$ then $K(z,u^{(z)})(y) = K(x,u)(y)$,

(iii) if $y \sqsubseteq z$ then $K(x,u)(z)_{(y)} = K(x,u)(y)$,

(iv) $K(x, \bot_{F(x)})(y) = \bot_{F(y)}$.

$K$ is a *transporter* if it is an $x,y$–transporter for all $x,y \in D$.

*Lemma 5.* Let $K \in \mathcal{K}(D,F)$ be an $x,y$–transporter. Then for all $u \in F(x)$ and $z \in D$:

(a) If $x \sqsubseteq y$ then $K(x,u)(y) = u^{(y)}$.

(b) If $y \sqsubseteq x$ then $K(x,u)(y) = u_{(y)}$.

(c) If $y \sqsubseteq x$ then $K(y,u_{(y)})(z) \sqsubseteq K(x,u)(z)$.

(d) If $z \sqsubseteq y$ then $K(x,u)(z)^{(y)} \sqsubseteq K(x,u)(y)$.

*Proof.* Easy.

*Definition 12.* Let $F: D \to \mathrm{DOM}$ and $G: \Sigma(D,F) \to \mathrm{DOM}$ be parametrizations. We define

$$\Gamma_{\Pi}^{D,F,G} \in [\mathcal{K}(D,F) \times \mathcal{K}(\Sigma(D,F),G) \to \mathcal{K}(D,\Pi(F,G))],$$

$$\Gamma_{\Sigma}^{D,F,G} \in [\mathcal{K}(D,F) \times \mathcal{K}(\Sigma(D,F),G) \to \mathcal{K}(D,\Sigma(F,G))]$$

by (omitting the superscripts $D,F,G$)

$$\Gamma_{\Pi}(K_1,K_2)(x,f)(y)(v) := K_2((x,K_1(y,v)(x)), f(K_1(y,v)(x)))(y,v),$$

$$\Gamma_{\Sigma}(K_1,K_2)(x,(u,r))(y) := (K_1(x,u)(y), K_2((x,u),r)(y,K_1(x,u)(y)))$$

where $K_1 \in \mathcal{K}(D,F)$, $K_2 \in \mathcal{K}(\Sigma(D,F),G)$, $x,y \in D$, $u \in F(x)$, $v \in F(y)$, $f \in \Pi(F,G)(x)$ and $r \in G(x,u)$. We will often write $\Gamma_{\Pi}$ and $\Gamma_{\Sigma}$ instead of $\Gamma_{\Pi}^{D,F,G}$ and $\Gamma_{\Sigma}^{D,F,G}$ provided $D$, $F$ and $G$ are clear from the context.

*Lemma 6.* Let $x,y \in D$, $K_1 \in \mathcal{K}(D,F)$ and $K_2 \in \mathcal{K}(\Sigma(D,F),G)$.

(a) If $K_1$ is a $y,x$–transporter and $K_2$ is an $(x,u),(y,v)$–transporter for all $u \in F(x)$ and $v \in F(y)$ then $\Gamma_{\Pi}(K_1,K_2) \in \mathcal{K}(D,\Pi(F,G))$ is an $x,y$–transporter.

(b) If $K_1$ is an $x,y$–transporter and $K_2$ is an $(x,u),(y,v)$–transporter for all $u \in F(x)$ and $v \in F(y)$ then $\Gamma_{\Sigma}(K_1,K_2) \in \mathcal{K}(D,\Sigma(F,G))$ is an $x,y$–transporter.

*Proof.* Easy. □

*Definition 13.* Let $E \in \mathrm{DOM}$ and $g \in [E \to D]$. Note that then $F \circ g : E \to \mathrm{DOM}$, defined by $(F \circ g)(i) := F(g(i))$ $(i \in E)$, is a parametrization. We define $\Delta_g^{E,D,F} \in [\mathcal{K}(D,F) \to \mathcal{K}(E, F \circ g)]$ by

$$\Delta_g^{E,D,F}(K)(i,u)(j) := K(g(i),u)(g(j))$$

where $K \in \mathcal{K}(D,F)$, $i,j \in E$ and $u \in F(g(i))$. Again we will often write $\Delta_g$ instead of $\Delta_g^{E,D,F}$.

*Lemma 7.* Let $E \in \mathrm{DOM}$, $i,j \in E$, $g \in [E \to D]$ and $K \in \mathcal{K}(D,F)$. If $K$ is a $g(i), g(j)$–transporter then $\Delta_g(K) \in \mathcal{K}(E, F \circ g)]$ is an $i,j$–transporter.

*Proof.* Easy. □

*Lemma 8.* If $K \in \mathcal{K}(D,F)$ is an $x_0, y_0$–transporter for all $x_0, y_0 \in D_0$ then $K$ is a transporter.

*Proof.* This follows easily from the following fact:

If $f, g \in \Pi(D,F)$ such that $f(x_0) \sqsubseteq g(x_0)$ for all $x_0 \in D_0$ then $f(x) \sqsubseteq g(x)$ for all $x \in D$. □

*Definition 14.* A *strong parametrization* is a parametrization $F: D \to \mathrm{DOM}$ together with a transporter $K_F \in \mathcal{K}(D,F)$ (the transporter associated with $F$, as we will say). If $G: \Sigma(D,F) \to \mathrm{DOM}$ is a further strong parametrization with associated transporter $K_G \in \mathcal{K}(\Sigma(D,F),G)$ then by lemma 6 the parametrizations $\Pi(F,G): D \to \mathrm{DOM}$ and $\Sigma(F,G): D \to \mathrm{DOM}$ become strong parametrizations by associating with them the transporters

$$\begin{aligned} K_{\Pi(F,G)} &:= \Gamma_\Pi(K_F, K_G) \in \mathcal{K}(D, \Pi(F,G)) \quad \text{and} \\ K_{\Sigma(F,G)} &:= \Gamma_\Sigma(K_F, K_G) \in \mathcal{K}(D, \Sigma(F,G)), \end{aligned}$$

respectively. Similarly, for $g \in [E \to D]$, $F \circ g : E \to \mathrm{DOM}$ becomes a strong parametrization by

$$K_{F \circ g} := \Delta_g(K_F) \in \mathcal{K}(E, F \circ g).$$

## 6. Uniform density and co–density

From now on we will assume that $F: D \to \mathrm{DOM}$ and $G: \Sigma(D,F) \to \mathrm{DOM}$ are strong parametrizations.

*Definition 15.* Let $x_0 \in D_0$. We say that $t_1, \ldots, t_k \in [\Sigma(D,F) \to \mathbb{B}^\perp]$ *separate* $u_1, \ldots, u_k \in F(x_0)_0$, if

(s1) for all $i \in \{1, \ldots, k\}$ and all $(x,u) \in \Sigma(D,F)$, if $u_i \sqsubseteq K_F(x,u)(x_0)$ then $t_i(x,u) = \#\mathrm{t}$, and

(s2) $\bigcap_{i=1}^k t_i^{-1}[\#\mathrm{t}] = \emptyset$.

Clearly, this is possible only if $u_1, \ldots, u_k$ are inconsistent in $F(x_0)$.

*Definition 16.* Let $L \subseteq D$ and $M \subseteq_L F$.

$M \subseteq_L F$ is *dense at* $x_0 \in D_0$ if

$$\forall u_0 \in F(x_0)_0 \exists d \in \Pi(L,M) \,.\, K_F(x_0,u_0) \sqsubseteq d.$$

$M \subseteq_L F$ is *co–dense at* $x_0 \in D_0$ if

$$\forall \vec{u} \in F(x_0)_0 \text{ incons.} \exists \vec{t} \in [\Sigma_*(L,M) \to_* \mathbb{B}] \,.\, \vec{t} \text{ separate } \vec{u}.$$

$M \subseteq_L F$ is *dense* respectively *co–dense* if it is dense respectively co–dense at all $x_0 \in D_0$.

If $G\colon \Sigma(D,F) \to \mathrm{DOM}$ is a strong parametrization, $L \subseteq \Sigma(D,F)$ and $N \subseteq_L G$ we say that $N$ is *dense* respectively *co–dense* at $x_0 \in D_0$ if it is dense respectively co–dense at $(x_0,u_0)$ for all $u_0 \in F(x_0)_0$.

*Lemma 9.* If $M \subseteq_L F$ is dense respectively co–dense then for each $x \in L$ the set $M(x) \subseteq F(x)$ is dense respectively co–dense in the sense of [1].

*Proof.* Easy. Definition 11 (i) is needed. □

*Definition 17.* Let $x_0 \in D_0$. We say that $t_1, \ldots, t_k \in [\Sigma(D,F) \to \mathbb{B}^\perp]$ *simultaneously separate inconsistent subsets of* $u_1, \ldots, u_k \in F(x_0)_0$ if

(s1) for all $i \in \{1, \ldots, k\}$ and all $(x,u) \in \Sigma(D,F)$,
if $u_i \sqsubseteq K_F(x,u)(x_0)$ then $t_i(x,u) = \#\mathrm{t}$, and

($\tilde{\mathrm{s2}}$) for all $J \subseteq \{1, \ldots, k\}$,
if $\{u_i : i \in J\}$ is inconsistent then $\bigcap_{i \in J} t^{-1}[\#\mathrm{t}] = \emptyset$.

*Lemma 10.* Let $M \subseteq_L F$ be co–dense at $x_0 \in D_0$. Then for each $u_1, \ldots, u_k \in F(x_0)_0$ there are $t_1, \ldots, t_k \in [\Sigma_*(L,M) \to_* \mathbb{B}]$ simultaneously separating inconsistent subsets of $u_1, \ldots, u_k$.

*Proof.* Define

$$\mathcal{J} := \{J \subseteq \{1, \ldots, k\} : \{u_i : i \in J\} \text{ is inconsistent in } F(x_0)\}.$$

Choose for each $J \in \mathcal{J}$ tests $t_{J,i} \in [\Sigma_*(L,M) \to_* \mathbb{B}]$ $(i \in J)$ separating the $u_i$ $(i \in J)$. For $i \in \{1,\dots,k\}$ define now

$$t_i := \bigwedge\{t_{J,i} : J \in \mathcal{J},\, i \in J\},$$

i.e. $t_i(x,u) = \#\mathrm{t}$ if $t_{J,i}(x,u) = \#\mathrm{t}$ for all $i \in J \in \mathcal{J}$, $t_i(x,u) = \#\mathrm{f}$ if $t_{J,i}(x,u) = \#\mathrm{f}$ for some $i \in J \in \mathcal{J}$ and $t_i(x,u) = \bot$ otherwise. It is easy to verify that $t_1,\dots,t_k$ simultaneously separate inconsistent subsets of $u_1,\dots,u_k$. Compare also with [1]. □

## 7. Density for Π and Σ

The following theorem contains the main result of this paper. It generalizes Kleene's and Kreisel's density theorems [6],[7], and is also the key to the density theorem for universes (theorem 3) generalizing Normann's density theorem [11].

*Theorem 2.* Let $F\colon D \to \mathrm{DOM}$ and $G\colon \Sigma(D,F) \to \mathrm{DOM}$ be strong parametrizations and let $L \subseteq D$, $M \subseteq_L F$, and $N \subseteq_{\Sigma_*(L,M)} G$. Recall that then $\Pi(N,M) \subseteq_L \Pi(F,G)$ and $\Sigma(N,M) \subseteq_L \Sigma(F,G)$ and that $\Pi(F,G)$ and $\Sigma(F,G)$ are strong parametrizations by lemma 6 and definition 14. Let $a_0 \in D_0$.

(1) If $M$ is co–dense at $a_0$ and $N$ is dense at $a_0$ then $\Pi_*(M,N)$ is dense at $a_0$.

(2) If $M$ is dense at $a_0$ and $N$ is co–dense at $a_0$ then $\Pi_*(M,N)$ is co–dense at $a_0$.

(3) If $M$ is dense at $a_0$ and $N$ is dense at $a_0$ then $\Sigma_*(M,N)$ is dense at $a_0$.

(4) If $M$ is co–dense at $a_0$ and $N$ is co–dense at $a_0$ then $\Sigma_*(M,N)$ is co–dense at $a_0$.

*Proof.* (1) Let $f_0 \in \Pi(F,G)(a_0)_0$, say

$$f_0 = \bigsqcup_{i=1}^{k} \langle b_i, c_i\rangle$$

for some $b_i \in F(a_0)_0$ and $c_i \in G(a_0,b_i)_0$. By lemma 10 there are tests $t_1,\dots,t_k \in [\Sigma_*(L,M) \to_* \mathbb{B}]$ simultaneously separating inconsistent subsets of $b_1,\dots,b_k$. Let $\mathcal{J} := \{J \subseteq \{1,\dots,k\} : \{b_i : i \in J\}$ is consistent in $F(a_0)\}$. For $J \in \mathcal{J}$ define

$$b_J := \bigsqcup_{i\in J} b_i \quad\text{and}\quad c_J := \bigsqcup_{i\in J} c_i^{(b_J)}.$$

Since $N$ is dense at $a_0$, for each $J \in \mathcal{J}$ there is $d_J \in \Pi(\Sigma_*(L,M),N)$ such that

$$(+) \qquad K_G((a_0,b_J),c_J) \sqsubseteq d_J.$$

For $(x,u) \in \Sigma(D,F)$ we let $J(x,u) := \{i \in \{1,\ldots,k\} : t_i(x,u) = \#\mathrm{t}\}$. By the choice of the $t_i$ $J(x,u) \in \mathcal{J}$ and hence $d_{J(x,u)}$ is defined. Let furthermore

$$U := \bigcap_{i=1}^{k} t_i^{-1}[\mathbb{B}]$$

which is an open subset of $\Sigma(D,F)$. Now for $x \in D$ and $u \in F(x)$ we define

$$d(x)(u) := \begin{cases} d_{J(x,u)}(x,u) & \text{if } (x,u) \in U \\ K_{\Pi(F,G)}(a_0,f_0)(x)(u) & \text{otherwise.} \end{cases}$$

In order to prove that this works we first show that

$$(*) \qquad K_{\Pi(F,G)}(a_0,f_0)(x)(u) \sqsubseteq d_{J(x,u)}(x,u)$$

for all $(x,u) \in \Sigma(D,F)$.

Proof of $(*)$: Let $(x,u) \in \Sigma(D,F)$ and let $J := \{i \in \{1,\ldots,k\} : b_i \sqsubseteq K_F(x,u)(a_0)\}$. Then $f_0(K_F(x,u)(a_0)) = c_J{}^{(a_0,K_F(x,u)(a_0))}$. Furthermore, by the choice of the $t_i$ we have $J \subseteq J(x,u)$. Hence, by definition 11 (ii) and monotonicity of $K_G$ as well as $(+)$, we have

$$\begin{aligned} K_{\Pi(F,G)}(a_0,f_0)(x)(u) &= K_G((a_0,K_F(x,u)(a_0)),f_0(K_F(x,u)(a_0)))(x,u) \\ &= K_G((a_0,K_F(x,u)(a_0)),c_J{}^{(a_0,K_F(x,u)(a_0))})(x,u) \\ &= K_G((a_0,b_J),c_J)(x,u) \\ &\sqsubseteq K_G((a_0,b_{J(x,u)}),c_{J(x,u)})(x,u) \\ &\sqsubseteq d_{J(x,u)}(x,u) \end{aligned}$$

which proves $(*)$. From $(*)$ and the fact that $U$ is open it follows immediately that $d$ is continuous, i.e. $d \in \Pi(D,\Pi(F,G))$ (note that if $(x,u),(\tilde{x},\tilde{y}) \in U$ and $(x,u) \sqsubseteq (\tilde{x},\tilde{y})$ then $J(x,u) = J(\tilde{x},\tilde{y})$). Furthermore, by $(*)$, $K_{\Pi(F,G)}(a_0,f_0) \sqsubseteq d$ and clearly $d \in \Pi_*(L,\Pi_*(M,N))$.

(2) Let $f_1,\ldots,f_k \in \Pi(F,G)(a_0)_0$ be inconsistent. Clearly there exists some $b_0 \in F(a_0)_0$ such that $\{f_1(b_0),\ldots,f_k(b_0)\} \subseteq G(a_0,b_0)_0$ is inconsistent. Since $M$ is dense at $a_0$ there is some $d \in \Pi_*(L,M)$ such that

$$K_F(a_0,b_0) \sqsubseteq d.$$

Since the totality $N$ is co–dense at $(a_0,b_0)$, there are tests $t_1,\ldots,t_k \in [\Sigma_*(\Sigma_*(L,M),N) \to_* \mathbb{B}]$ separating $f_1(b_0),\ldots,f_k(b_0)$. We define the tests $\tilde{t}_1,\ldots,\tilde{t}_k \in [\Sigma(D,\Pi(F,G)) \to \mathbb{B}^\perp]$ by

$$\tilde{t}_i(x,f) := t_i((x,d(x)),f(d(x))).$$

Clearly $\tilde{t}_i \in [\Sigma_*(L, \Pi_*(M,N)) \to_* \mathbb{B}]$. In order to verify that $\tilde{t}_1, \dots, \tilde{t}_k$ separate $f_1, \dots, f_k$ assume first $f_i \sqsubseteq K_{\Pi(F,G)}(x,f)(a_0)$. For proving $\tilde{t}_i(x,f) = \#\mathrm{t}$ it suffices to show $f_i(b_0) \sqsubseteq K_G((x,d(x)), f(d(x)))(a_0,b_0)$. We have

$$\begin{aligned} f_i(b_0) &\sqsubseteq K_{\Pi(F,G)}(x,f)(a_0)(b_0) \\ &= K_G((x, K_F(a_0,b_0)(x)), f(K_F(a_0,b_0)(x)))(a_0,b_0) \\ &\sqsubseteq K_G((x,d(x)), f(d(x)))(a_0,b_0). \end{aligned}$$

Clearly $\bigcap_{i=1}^k \tilde{t}_i^{-1}[\#\mathrm{t}] = \emptyset$.

(3) Let $(b_0,c_0) \in \Sigma(F,G)(a_0)_0$, i.e. $b_0 \in F(a_0)_0$ and $c_0 \in G(a_0,b_0)_0$. Since $M$ and $N$ are dense at $a_0$ there are $d_1 \in \Pi_*(L,M)$ and $d_2 \in \Pi_*(\Sigma_*(L,M),N)$ such that

$$K_F(a_0,b_0) \sqsubseteq d_1 \quad \text{and} \quad K_G((a_0,b_0),c_0) \sqsubseteq d_2.$$

Define $d \in \Pi(D, \Sigma(F,G))$ by

$$d(x) := (d_1(x), d_2(x, d_1(x))).$$

Clearly $d \in \Pi_*(L, \Sigma_*(M,N))$. Furthermore

$$\begin{aligned} & K_{\Sigma(F,G)}(a_0,(b_0,c_0))(x) \\ = {} & (K_F(a_0,b_0)(x), K_G((a_0,b_0),c_0)(x, K_F(a_0,b_0)(x))) \\ \sqsubseteq {} & (d_1(x), d_2(x,d_1(x))) = d(x). \end{aligned}$$

(4) Let $\{(b_1,c_1), \dots, (b_k,c_k)\} \subseteq \Sigma(F,G)(a_0)_0$ be inconsistent. There are two cases.

*Case 1*: $\{b_1, \dots, b_k\} \subseteq F(a_0)_0$ is inconsistent. Let the total tests $t_1, \dots, t_k \in [\Sigma_*(L,M) \to_* \mathbb{B}]$ separate $b_1, \dots, b_k$. Define $\tilde{t}_1, \dots, \tilde{t}_k \in [\Sigma(D, \Sigma(F,G)) \to \mathbb{B}^\perp]$ by

$$\tilde{t}_i(x,(u,r)) := t_i(x,u).$$

Clearly $\tilde{t}_i \in [\Sigma_*(L, \Sigma_*(M,N)) \to_* \mathbb{B}]$. It's easy to see that $\tilde{t}_1, \dots, \tilde{t}_k$ separate $(b_1,c_1), \dots, (b_k,c_k)$.

*Case 2*: $b_0 := \bigsqcup_{i=1}^k b_i$ exists. Then $\{c_1^{(a_0,b_0)}, \dots, c_k^{(a_0,b_0)}\} \subseteq G(a_0,b_0)_0$ is inconsistent. Let $t_1, \dots, t_k \in [\Sigma_*(\Sigma_*(L,M),N) \to_* \mathbb{B}]$ separate $c_1^{(a_0,b_0)}, \dots, c_k^{(a_0,b_0)}$. Define $\tilde{t}_1, \dots, \tilde{t}_k \in [\Sigma(D, \Sigma(F,G)) \to \mathbb{B}^\perp]$ by

$$\tilde{t}_i(x,(u,r)) := t_i((x,u),r).$$

Clearly $\tilde{t}_i \in [\Sigma_*(L, \Sigma_*(M,N)) \to_* \mathbb{B}]$. In order to prove that $\tilde{t}_1, \dots, \tilde{t}_k$ separate $(b_1,c_1), \dots, (b_k,c_k)$ assume $(b_i,c_i) \sqsubseteq K_{\Sigma(F,G)}(x,(u,r))(a_0)$, i.e.

$$b_i \sqsubseteq K_F(x,u)(a_0) \quad \text{and} \quad c_i \sqsubseteq K_G((x,u),r)(a_0, K_F(x,u)(a_0))_{(a_0,b_i)}.$$

Hence, by definition 11 (iii), $c_i \sqsubseteq K_G((x,u),r)(a_0,b_i)$ and lemma 5 (d),

$$c_i^{(a_0,b_0)} \sqsubseteq K_G((x,u),r)(a_0,b_i)^{(a_0,b_0)} \sqsubseteq K_G((x,u),r)(a_0,b_0).$$

Therefore $\tilde{t}_i(x,(u,r)) = t_i((x,u),r) = \#\mathrm{t}$. Clearly the tests $\tilde{t}_i$ are total. □

*Lemma 11.* Let $F: D \to \mathrm{DOM}$ be a strong parametrization, $L \subseteq D$, $M \subseteq_L F$, $J \subseteq E$ and $g \in [J \to L]$. Recall that then $M \circ g \subseteq_J F \circ g$ and that $F \circ g: E \to \mathrm{DOM}$ is a strong parametrization by lemma 7. Let $i_0 \in E_0$ such that $g(i_0) \in D_0$.

(1) If $M$ is dense at $g(i_0)$ then $M \circ g$ is dense at $i_0$.

(2) If $M$ is co–dense at $g(i_0)$ then $M \circ g$ is co–dense at $i_0$.

*Proof.* (1) Let $u_0 \in F(g(i_0))_0$. Since $M$ is dense at $g(i_0)$ there is some $d \in \Pi_*(L,M)$ such that $K_F(g(i_0),u_0) \sqsubseteq d$. Then clearly $d \circ g \in \Pi_*(J, M \circ g)$ and

$$K_{F \circ g}(i_0,u_0)(i) = K_F(g(i_0),u_0)(g(i)) \sqsubseteq d(g(i)) = (d \circ g)(i)$$

for all $i \in E$.

(2) Let $u_1,\ldots,u_k \in F(g(i_0))$ be separable. Since $M$ is co–dense at $g(i_0)$ there are $t_1,\ldots,t_k \in [\Sigma_*(L,M) \to_* \mathbb{B}]$ separating $u_1,\ldots,u_k$. Clearly $t_1 \circ g,\ldots,t_k \circ g$ are in $[\Sigma_*(J, M \circ g) \to_* \mathbb{B}]$ and they separate $u_1,\ldots,u_k$. □

## 8. Perfect quantifiers and operators

In order to obtain a general density theorem for universes closed under $\Pi$ and $\Sigma$ and other operators $\Phi: \mathrm{PAR} \to \mathrm{DOM}$ we isolate the properties of $\Pi$ and $\Sigma$ which are essential for such a theorem. It turns out that we have to formulate these properties also for functors $\Psi: \mathrm{PAR} \to \mathrm{PAR}$.

*Definition 18.* If $F: D \to \mathrm{DOM}$ and $G: \Sigma(D,F) \to \mathrm{DOM}$ are parametrizations then we call $G$ a *parametrization over* $F$, or $(F,G)$ a *2–parametrization.* Instead of $(F,G)$ we will sometimes write more explicitely $(D,F,G)$. The collection of 2–parametrizations becomes a category 2–PAR with morphisms

$$(\eta,\sigma,\tau): (D,F,G) \to (D_1,F_1,G_1)$$

where

$$\begin{aligned}(\eta, \sigma) &: (D, F) \to (D_1, F_1) \text{ and} \\ (\Sigma(\eta, \sigma), \tau) &: (\Sigma(D, F), G) \to (\Sigma(D_1, F_1), G_1)\end{aligned}$$

are morphisms in PAR. Composition is defined by

$$(\eta_1, \sigma_1, \tau_1) \circ (\eta, \sigma, \tau) = (\eta_2, \sigma_2, \tau_2),$$

where

$$\begin{aligned}\eta_2 &= \eta_1 \circ \eta, \\ \sigma_2 &= \lambda x.\sigma_1(\eta(x)) \circ \sigma(x) \text{ and} \\ \tau_2 &= \lambda(x, u).\tau_1(\eta(x), \sigma(x)(u)) \circ \tau(x, u)).\end{aligned}$$

$(F, G)$ is a *strong 2–parametrization* if $F$ and $G$ are strong parametrizations.

A *totality on (F,G)* is a triple $(L, M, N)$ such that

$$M \subseteq_L F \quad \text{and} \quad N \subseteq_{\Sigma_*(L,M)} G.$$

*Definition 19.* Let $\Phi\colon \mathrm{PAR} \to \mathrm{DOM}$ be a continuous functor. We define a continuous functor (overloading names) $\Phi\colon 2\text{–PAR} \to \mathrm{PAR}$ by

$$\mathrm{dom}(\Phi(F, G)) = \mathrm{dom}(F),$$

$$\Phi(F, G)(x) = \Phi(F(x), \lambda u.G(x, u)),$$

$$\Phi(F, G)[x, y] = \Phi(F[x, y], \lambda u.G[(x, u), (y, u^{(y)})]).$$

$$\Phi(\eta, \sigma, \tau) = (\eta, \lambda x.\Phi(\sigma(x), \lambda u.\tau(x, u))).$$

Similarly for a continuous functor $\Psi\colon \mathrm{PAR} \to \mathrm{PAR}$ we define a continuous functor (again overloading names) $\Psi\colon 2\text{–PAR} \to \mathrm{PAR}$ by

$$\mathrm{dom}(\Psi(F, G)) = (\Sigma x \in \mathrm{dom}(F))\mathrm{dom}(\Psi(F(x), \lambda u.G(x, u)),$$

$$\Psi(F, G)(x, a) = \Psi(F(x), \lambda u.G(x, u))(a),$$

$$\Psi(F, G)[(x, a), (y, b)] = \Psi(F(x), \lambda u.G(x, u))[\eta_{x,y}(a), b] \circ \tau_{x,y}(a)$$
where $(\eta_{x,y}, \tau_{x,y}) = \Psi(F[x, y], \lambda u.G[(x, u), (y, u^{(y)})])$.

$\Psi(\eta, \sigma, \tau)$ can be defined similarly.

These definitions extend to totalities on $\Phi$ and $\Psi$ in the obvious way.

Note that this is in harmony with the definition of $\Pi(F, G)$ and $\Sigma(F, G)$ in section 2.

*Definition 20.* Let $\mathcal{C}$ be a category and $\Phi: \mathcal{C} \to \mathrm{DOM}$ a continuous functor. A *continuous section* of $\Phi$ is a map $\Gamma$ assigning to every $a \in \mathcal{C}$ some $\Gamma(a) \in \Phi(a)$ such that

(i) if $f \in \mathcal{C}[a, b]$ then $\Phi(f)(\Gamma(a)) \sqsubseteq \Gamma(b)$,

(ii) if $(a, f_i)$ is a directed co–limit of $(a_i, f_{ij})$ in $\mathcal{C}$ then

$$\Gamma(a) = \bigsqcup_i \Phi(f_i)(\Gamma(a_i)).$$

Instead of $\Gamma(a)$ we will often write $\Gamma^a$. Now assume in addition that $(\mathcal{C}, \mathcal{C}_*)$ is a category with totality and $\Phi_*$ is a totality on $\Phi$. Then $\Gamma$ is called *total* if for every $a \in \mathcal{C}$ and every $a_* \in \mathcal{C}_*(a)$ we have that $\Gamma(a) \in \Phi_*(a_*)$.

*Example.* Consider $\Phi: \mathrm{DOM} \to \mathrm{DOM}$, $\Phi(D) := [D \to D]$ and the continuous section id defined by $\mathrm{id}(D) := \lambda x \in D.x$. Recall that by definition 6 DOM is a category with totality. Let $\Phi_*$ be the standard totality on $\Phi$, namely $\Phi_*(M) = [M \to_* M]$ for $M \subseteq D \in \mathrm{DOM}$. Then id is total since $\mathrm{id}^D \in [M \to_* M]$ for all $M \subseteq D$.

*Definition 21.* (i) Let $F: D \to \mathrm{DOM}$ be a parametrization. $K \in \mathcal{K}(F)$ is a *transporter for $F$ at $x, y \in D$* if $K$ is an $x, y$– and a $y, x$–transporter for $F$.

(ii) Let $(D, F, G) \in 2\text{–PAR}$. A *transporter for $(F, G)$ at $x, y \in D$* is a pair $(K_1, K_2)$ such that $K_1$ is a transporter for $F$ at $x, y$ and $K_2$ is a transporter for $G$ at $(x, u), (y, v)$ for all $u \in F(x)$ and $v \in F(y)$.

(iii) Let $\Phi: \mathrm{PAR} \to \mathrm{DOM}$ (resp. $\Phi: \mathrm{PAR} \to \mathrm{PAR}$) be a continuous functor. A *transporter for $\Phi$* is a continuous section $\Gamma$ of

$$\lambda(D, F, G) \in 2\text{–PAR}\,.\,[\mathcal{K}(D, F) \times \mathcal{K}(\Sigma(D, F), G) \to \mathcal{K}(\Phi(F, G))]$$

(resp. $\lambda(D, F, G) \in 2\text{–PAR}\,.$
$[\mathcal{K}(D, F) \times \mathcal{K}(\Sigma(D, F), G) \to \mathcal{K}((\mathrm{dom} \circ \Phi)(F, G)) \times \mathcal{K}(\Phi(F, G))]$)
such that for all $x, y \in D$, $K_1 \in \mathcal{K}(D, F)$ and $K_2 \in \mathcal{K}(\Sigma(D, F), G)$, if $(K_1, K_2)$ is a transporter for $(F, G)$ at $x, y$ then $\Gamma(F, G)(K_1, K_2)$ is a transporter for $\Phi(F, G) \in \mathrm{PAR}$ (resp. $((\mathrm{dom} \circ \Phi)(F, G), \Phi(F, G)) \in 2\text{–PAR}$) at $x, y$ .

*Example.* In definition 14 we defined continuous sections $\Gamma_\Pi$ and $\Gamma_\Sigma$ which, by lemma 6, are transporters for $\Pi$ and $\Sigma$ respectively.

*Definition 22.* (i) Let $(D, F, G) \in$ 2–PAR be a strong 2–parametrization and $(L, M, N)$ a totality on $(D, F, G)$. $(L, M, N)$ is *uniformly dense and co–dense* at $x_0 \in D_0$ if $(L, M)$ and $(\Sigma_*(L, M), N)$ are both uniformly dense and uniformly co–dense at $x_0$.

(ii) Let $\Phi\colon \mathrm{PAR} \to \mathrm{DOM}$ (resp. $\Phi\colon \mathrm{PAR} \to \mathrm{PAR}$) be a continuous functor and $\Gamma$ a transporter for $\Phi$. A totality $\Phi_*$ on $\Phi$ is *uniformly dense and co–dense w.r.t.* $\Gamma$, if for every strong parametrization $(D, F, G) \in$ 2–PAR with transporters $K_1 \in \mathcal{K}(D, F)$, $K_2 \in \mathcal{K}(\Sigma(D, F), G)$, every $x_0 \in D_0$ and every totality $(L, M, N)$ on $(D, F, G)$ which is uniformly dense and co–dense at $x_0$ w.r.t. $K_1, K_2$ we have that $\Phi_*(L, M, N)$ (resp. $((\mathrm{dom}_* \circ \Phi_*)(L, M, N), \Phi_*(L, M, N)))$ is uniformly dense and co–dense at $x_0$ w.r.t. $\Gamma^{(D,F,G)}(K_1, K_2)$.

*Example.* By the density theorem 2 the standard totalities on $\Pi$ and $\Sigma$ are dense and co–dense w.r.t. the transporters $\Gamma_\Pi$ and $\Gamma_\Sigma$.

*Definition 23.* Let $\Phi\colon \mathrm{PAR} \to \mathrm{DOM}$ (resp. $\Phi\colon \mathrm{PAR} \to \mathrm{PAR}$) be a continuous functor. A totality $\Phi_*$ on $\Phi$ is *uniformly nonempty* if there is a continuous section $\mathrm{sel}_\Phi$ of

$$\lambda F \in \mathrm{PAR}\,.\,[\mathrm{dom}(F) \times \Pi(F) \to \Phi(F)]$$

$$(\text{resp. } \lambda F \in \mathrm{PAR}\,.\,[\mathrm{dom}(F) \times \Pi(F) \to \mathrm{dom}(\Phi(F)) \times \Pi(\Phi(F))]\ )$$

which is total w.r.t. the totality induced by $\Phi_*$. This means that if $M$ is a totality on $F$, $x \in \mathrm{dom}_*(M)$ and $f \in \Pi_*(M)$ then $\mathrm{sel}^F_\Phi x f \in \Phi_*(M)$ (resp. $\in \mathrm{dom}_*(\Phi_*(M)) \times \Pi_*(\Phi_*(M)))$.

*Example.* The standard totalities on $\Pi$ and $\Sigma$ are uniformly nonempty. Just let $\mathrm{sel}^{(D,F)}_\Pi x f := f$ and $\mathrm{sel}^{(D,F)}_\Sigma x f := (x, f(x))$.

*Definition 24.* A *perfect quantifier* (resp. a *perfect operator*) is a quadruple $(\Phi, \Phi_*, \Gamma_\Phi, \mathrm{sel}_\Phi)$ such that

(i) $\Phi\colon \mathrm{PAR} \to \mathrm{DOM}$ (resp. $\Phi\colon \mathrm{PAR} \to \mathrm{PAR}$) is a continuous functor,

(ii) $\Phi_*$ is a totality on $\Phi$,

(iii) $\Gamma_\Phi$ is a transporter for $\Phi$,

(iv) $\Phi_*$ is uniformly dense and co–dense w.r.t. $\Gamma_\Phi$,

(v) $\Phi_*$ is uniformly nonempty via $\mathrm{sel}_\Phi$.

Frequently we will denote a perfect quantifier or operator by its first component $\Phi$. We will also call $(\Phi, \Phi_*)$ perfect if there are $\Gamma_\Phi$ and $\mathrm{sel}_\Phi$ such that $(\Phi, \Phi_*, \Gamma_\Phi, \mathrm{sel}_\Phi)$ is perfect. Obviously if $\Psi\colon \mathrm{PAR} \to \mathrm{PAR}$ is perfect then $\mathrm{dom} \circ \Psi\colon \mathrm{PAR} \to \mathrm{DOM}$ is perfect, too.

*Example.* By the examples above we see that $(\Pi, \Pi_*, \Gamma_\Pi, \mathrm{sel}_\Pi)$ and $(\Sigma, \Sigma_*, \Gamma_\Sigma, \mathrm{sel}_\Sigma)$ are perfect quantifiers.

## 9. Density for universes

*Theorem 3.* If $\vec{\Phi}$ are perfect quantifiers then the universe operator $\mathcal{U}[\vec{\Phi}]\colon \mathrm{PAR} \to \mathrm{PAR}$ together with its wellfounded totality $\mathcal{U}_{\mathrm{wf}}[\vec{\Phi}_*]$ is a perfect operator.

*Proof.* Let $\mathcal{U} := \mathcal{U}[\vec{\Phi}]\colon \mathrm{PAR} \to \mathrm{PAR}$.

*1. $\mathcal{U}$ is uniformly nonempty.*

Using notations as in definition 4 we define for $(A, B) \in \mathrm{PAR}$, $a \in A$ and $g \in \Pi(A, B)$

$$\mathrm{sel}^0_{\mathcal{U}}(A, B)(a, g) = \alpha(a) \in S := \mathrm{dom}(\mathcal{U}(A, B)).$$

Clearly, if $a \in A_*$ and $g \in \Pi_*(A_*, B_*)$ then $\mathrm{sel}^0_{\mathcal{U}}(A, B)(a, g) \in S_{\mathrm{wf}}$. Given furthermore $s \in S$ we define $\mathrm{sel}^1_{\mathcal{U}}(A, B)(a, g)(s) = \mathrm{sel}(s) \in I(s) := \mathcal{U}(A, B)(s)$ where $\mathrm{sel}(s)$ is defined recursively by

$$\mathrm{sel}(s) = \begin{cases} a & \text{if } \tau(s) = \alpha, \\ g(b) & \text{if } \tau(s) = (\beta, b), \\ \mathrm{sel}_{\Phi_i}(I(s_1), I \circ f)(\mathrm{sel}(s_1), \mathrm{sel} \circ f) & \text{if } \tau(s) = (\varphi_i, s_1, f), \\ \mathrm{sel}^1_{\Psi_j}(I(s_1), I \circ f)(\mathrm{sel}(s_1), \mathrm{sel} \circ f)(u) & \text{if } \tau(s) = (\psi_j, s_1, f, u), \\ \bot & \text{if } \tau(s) = \bot. \end{cases}$$

By induction on the inductive definition of $S_{\mathrm{wf}}$ one easily shows that if $a \in A_*$ and $g \in \Pi_*(A_*, B_*)$ then $\mathrm{sel}^1_{\mathcal{U}}(A, B)(a, g)(s) \in I_{\mathrm{tot}}$. Hence

$$\mathrm{sel}_{\mathcal{U}} := \lambda(A, B)\lambda(a, g).(\mathrm{sel}^0_{\mathcal{U}}(A, B)(a, g), \mathrm{sel}^1_{\mathcal{U}}(A, B)(a, g))$$

is a total continuous section. This proves that $\mathcal{U}_{\mathrm{wf}}$ is uniformly nonempty.

It remains to define a transporter for $\mathcal{U}$ with respect to which $\mathcal{U}_{\mathrm{wf}}$ is uniformly dense and co–dense. To this end we fix a 2–parametrization $(E, A, B) \in$ 2–PAR, i.e. $(E, A), (\Sigma(E, A), B) \in \mathrm{PAR}$, and let $(E, S, I) := \mathcal{U}(E, A, B)$, i.e. $\mathrm{dom}(S) = E$, $S(e) = \mathrm{dom}\mathcal{U}(A(e), \lambda a.B(e, a))$ for $e \in E$, $\mathrm{dom}(I) = \Sigma(E, S)$ and $I(e, s) = \mathcal{U}(A(e), \lambda a.B(e, a))(s)$ for $s \in S(e)$.

We are essentially in the same situation as in definition 4 except that everything depends on an extra parameter $e \in E$. Therefore

$$S(e) \simeq A(e) + D^1 + \ldots + D^k,$$

where $D^i = (\Sigma s \in S(e))[I(e,s) \to S(e)]$. The isomorphism from left to right is given by $\tau$ satisfying

if $\tau(e,s) = \bot$ then $I(e,s) \simeq \{\bot\}$,

if $\tau(e,s) = (\beta, a)$ then $I(e,s) \simeq B(e,a)$,

if $\tau(e,s) = (\varphi_i, s_1, f)$ then $I(e,s) \simeq \Phi_i(I(e,s_1), \lambda x.I(e,f(x)))$.

The inverse of $\tau$ is given by injections $\alpha \in \Pi(E,S)$, $\beta \in (\Pi e \in E)[A(e) \to S(e)]$, $\varphi_i \in (\Pi e \in E)[(\Sigma s \in S(e))[I(e,s) \to S(e)] \to S(e)]$ and such that

$\tau(e, \beta(e)(a)) = (\beta, a)$ for $a \in A(e)$,

$\tau(e, \varphi_i(e)(s,f)) = (\varphi_i, s, f)$ for $s \in S(e)$ and $f \in [I(e,s) \to S(e)]$.

By $\tau_0(e,s)$ we denote the first component of $\tau(e,s)$, i.e. the label of $s$.

For $e \in E$ and compact $s_0 \in S(e)_0$ we have $\mathrm{rk}(e,s_0) \in \mathbb{N}$, such that if $\tau(e,s_0) = (\varphi_i, s_1, f)$ then $\mathrm{rk}(e,s_1) < \mathrm{rk}(e,s_0)$ and for all $x \in I(e,s_1)$, $f(x)$ is compact and $\mathrm{rk}(e,f(x)) < \mathrm{rk}(e,s_0)$.

Furthermore we define continuous functions $\delta\colon \Sigma(E,S) \to \Sigma(E,S)$ and $p\colon \Sigma(\Sigma(E,S), I \circ \delta) \to \Sigma(E,S)$ by $\delta(e,s) := (e,s_1)$ and $p(e,s,x) := (e,f(x))$ if $\tau(s) = (\varphi_i, s_1, f)$, and $\delta(e,s) = p(e,s,x) = \bot_S$ otherwise.

*2. Definition and verification of a transporter $\Gamma_{\mathcal{U}}$ for $\mathcal{U}$.*

Since $\Phi_i$ and $\Psi_j$ are perfect there are transporters $\Gamma_{\Phi_i}$ and $\Gamma_{\Psi_j}$ for $\Phi_i$ and $\Psi_j$ respectively. For $(D,F,G) \in$ 2–PAR we let

$$\Gamma_{\Psi_j}(F,G) =: (\Gamma^0_{\Psi_j}(F,G), \Gamma^1_{\Psi_j}(F,G)).$$

So, if $x, y \in D$, $K_1 \in \mathcal{K}(D,F)$ and $K_2 \in \mathcal{K}(\Sigma(D,F),G)$, if $(K_1,K_2)$ is a transporter for $(F,G)$ at $x,y$, then $\Gamma^0_{\Psi_j}(F,G)(K_1,K_2)$ is a transporter for $(\mathrm{dom} \circ \Psi_j)(F,G)$ at $x,y$, and $\Gamma^1_{\Psi_j}(F,G)(K_1,K_2)$ is a transporter for $\Psi_j(F,G)$ at $x,y$ .

We have to define $K_S \in \mathcal{K}(E,S)$ and $K_I \in \mathcal{K}(\Sigma(E,S),I)$ 'continuously' from $(E,A,B)$, $K_A \in \mathcal{K}(E,A)$ and $K_B \in \mathcal{K}(\Sigma(E,A),B)$.

First we define $K_I \in [\Sigma(\Sigma(E,S),I) \to \Pi(\Sigma(E,S),I)]$ recursively by

$K_I((e,s),x)(\tilde{e},\tilde{s}) :=$

$$\begin{cases} K_A(e,x)(\tilde{e}) & \text{if } \tau(e,s)=\tau(\tilde{e},\tilde{s})=\alpha, \\ K_B((e,a),x)(\tilde{e},\tilde{a}) & \text{if } \tau(e,s)=(\beta,a) \\ & \text{and } \tau(\tilde{e},\tilde{s})=(\beta,\tilde{a}), \\ \Gamma_{\Phi_i}(\Delta_\delta(K_I),\Delta_p(K_I))((e,s),x)(\tilde{e},\tilde{s}) & \text{if } \tau_0(e,s)=\tau_0(\tilde{e},\tilde{s})=\varphi_i, \\ \perp_{I(\tilde{e},\tilde{s})} & \text{otherwise} \end{cases}$$

Using $K_I$ we define $K_S \in [\Sigma(E,S) \to \Pi(E,S)]$ recursively by

$K_S(e,s)(\tilde{e}) :=$

$$\begin{cases} \alpha(\tilde{e}) & \text{if } \tau(e,s)=\alpha, \\ \beta(\tilde{e},K_A(e,a)(\tilde{e})) & \text{if } \tau(e,s)=(\beta,a), \\ \varphi_i(\tilde{e})(\tilde{s}_1,\tilde{f}) & \text{if } \tau(e,s)=(\varphi_i,s_1,f) \text{ where} \\ & \tilde{s}_1 := K_S(e,s_1)(\tilde{e}) \text{ and} \\ & \tilde{f} := \lambda\tilde{x}.K_S(e,f(K_I((\tilde{e},\tilde{s}_1),\tilde{x})(e,s_1)))(\tilde{e}) \\ \perp_{S(\tilde{e})} & \text{otherwise} \end{cases}$$

*Verification of $K_I$.*

Assume that $(K_A,K_B)$ is a transporter for $(A,B)$ at $e,\tilde{e} \in E$. We have to show that $K_I$ is a transporter for $(\Sigma(E,S),I)$ at $e,\tilde{e}$, i.e. $K_I$ is a transporter at $(e,s),(\tilde{e},\tilde{s})$ for all $s \in S(e)$ and $\tilde{s} \in S(\tilde{e})$.

First we show that $K_I$ has property (iv), i.e. $K_I((e,s),\perp_{I(e,s)})(\tilde{e},\tilde{s}) = \perp_{I(\tilde{e},\tilde{s})}$ for all $(e,s),(\tilde{e},\tilde{s}) \in \Sigma(E,S)$.

$K$ is defined as the least fixed point of a continuous function

$$\Phi\colon \mathcal{K}(\Sigma(E,S)S,I) \to \mathcal{K}(\Sigma(E,S),I),$$

i.e. $K = \bigsqcup_n \Phi^n(\perp_{\mathcal{K}(\Sigma(E,S),I)})$. One easily proves that

$$\Phi^n(\perp_{\mathcal{K}(S,I)})((e,s),\perp_{I(e,s)})(\tilde{e},\tilde{s}) = \perp_{I(\tilde{e},\tilde{s})}$$

for all $(e,s),(\tilde{e},\tilde{s}) \in \Sigma(E,S)$ by induction on $n$. Hence

$$K_I((e,s),\perp_{I(e,s)})(\tilde{e},\tilde{s}) = \bigsqcup_n \Phi^n(\perp_{\mathcal{K}(S,I)})((e,s),\perp_{I(e,s)})(\tilde{e},\tilde{s}) = \perp_{I(\tilde{e},\tilde{s})}.$$

By lemma 8 it suffices to show that $K_I$ is an $(e_0,s_0),(\tilde{e}_0,\tilde{s}_0)$–transporter for all $(e_0,s_0),(\tilde{e}_0,\tilde{s}_0) \in \Sigma(E,S)_0$. We prove this by induction on the maximum of $\mathrm{rk}(e_0,s_0)$ and $\mathrm{rk}(\tilde{e}_0,\tilde{s}_0)$.

*Case $\tau(e_0,s_0)=(\beta,a)$, $\tau(\tilde{e}_0,\tilde{s}_0)=(\beta,\tilde{a})$.*
Then $K_I((e_0,s_0),x)(\tilde{e}_0,\tilde{s}_0) = K_B((e_0,a),x)(\tilde{e}_0,\tilde{a})$.

(i) If $(e_0,s_0)=(\tilde{e}_0,\tilde{s}_0)$ then $a=\tilde{a}$ and hence

$$K_I((e_0,s_0),x)(\tilde{e}_0,\tilde{s}_0) = K_B((e_0,a),x)(\tilde{a}) = x.$$

(ii) If $(e_0, s_0) \sqsubseteq (e', s')$ then $\tau(e', s') = (\beta, b)$ where $a \sqsubseteq b$. Moreover $x^{(e',s')} = x^{(b)}$. Hence

$$K_I(e', s', x^{(e',s')})(\tilde{e}_0, \tilde{s}_0) = K_B(b, x^{(b)})(\tilde{a}) = K_B(a, x)(b).$$

(iii) If $\tilde{s}_0 \sqsubseteq (e', s')$ then $\tau(e', s') = (\beta, b)$ where $\tilde{a} \sqsubseteq b$. Hence

$$K_I(s_0, x)(e', s')_{(\tilde{e}_0, \tilde{s}_0)} = K_B(a, x)(b)_{(\tilde{a})} = K_B(a, x)(\tilde{a}).$$

*Case* $\tau_0(e_0, s_0) = \tau_0(\tilde{e}_0, \tilde{s}_0) = \varphi_i$. Then we have $\mathrm{rk}(\delta(e_0, s_0)) < \mathrm{rk}(e_0, s_0)$ and $\mathrm{rk}(\delta(\tilde{e}_0, \tilde{s}_0)) < \mathrm{rk}(\tilde{e}_0, \tilde{s}_0)$.

Hence, by i.h. $K_I$ is a $\delta(\tilde{e}_0, \tilde{s}_0), \delta(e_0, s_0)$–transporter and, by lemma 7, $\Delta_\delta(K_I)$ is an $(\tilde{e}_0, \tilde{s}_0), (e_0, s_0)$–transporter.

Furthermore for all $x \in I(\delta(e_0, s_0))$ and $\tilde{x} \in I(\delta(\tilde{e}_0, \tilde{s}_0))$ we have $\mathrm{rk}(p(e_0, s_0, x)) < \mathrm{rk}(e_0, s_0)$ and $\mathrm{rk}(p(\tilde{e}_0, \tilde{s}_0), \tilde{x})) < \mathrm{rk}(\tilde{e}_0, \tilde{s}_0)$.

Hence, by i.h. $K_I$ is a $p((e_0, s_0), x), p((\tilde{e}_0, \tilde{s}_0), \tilde{x})$–transporter and, by lemma 7, $\Delta_p(K_I)$ is an $((e_0, s_0), x), ((\tilde{e}_0, \tilde{s}_0), \tilde{x})$–transporter. Since $\Gamma_{\Phi_i}$ is a transporter for $\Phi_i$, $\Gamma_{\Phi_i}(\Delta_\delta(K_I), \Delta_p(K_I))$ is an $(e_0, s_0), (\tilde{e}_0, \tilde{s}_0)$–transporter. Since for all $((e', s'), (\tilde{e}', \tilde{s}')) \sqsupseteq ((e_0, s_0), (\tilde{e}_0, \tilde{s}_0))$ and all $y \in I(e', s')$ we have

$$K_I((e', s'), y)(\tilde{e}', \tilde{s}') = \Gamma_\Pi(\Delta_\delta(K_I), \Delta_p(K_I))(t, y)(\tilde{e}', \tilde{s}'),$$

it follows that $K_I$ is an $(e_0, s_0), (\tilde{e}_0, \tilde{s}_0)$–transporter.

*Case "otherwise".* Then $K_I((e_0, s_0), x)(\tilde{e}_0, \tilde{s}_0) = \bot_{I(\tilde{e}_0, \tilde{s}_0)}$.

(i) If $(e_0, s_0) = (\tilde{e}_0, \tilde{s}_0)$ then $\tau(e_0, s_0) = \bot$ and $I(e_0, s_0) = \{\bot\}$. Hence, if $x \in I(e_0, s_0)$ then $x = \bot = K_I((e_0, s_0), x)(\tilde{e}_0, \tilde{s}_0)$.

(ii) Assume $(e_0, s_0) \sqsubseteq (e', s')$ and let $x \in I(e_0, s_0)$. If $\tau(e_0, s_0) = \bot$ then $x = \bot_{I(e_0, s_0)}$ and hence, since we already proved that the strictness property (iv) holds,

$$\begin{aligned} K_I((e', s'), x^{(e',s')})(\tilde{e}_0, \tilde{s}_0) &= K_I((e', s'), \bot_{I(e',s')})(\tilde{e}_0, \tilde{s}_0) \\ &= \bot_{(\tilde{e}_0, \tilde{s}_0)} \\ &= K_I((e_0, s_0), x)(\tilde{e}_0, \tilde{s}_0). \end{aligned}$$

If $\tau(e_0, s_0) \neq \bot$ then $\tau_0(e', s') = \tau_0(e_0, s_0) \neq \tau_0(\tilde{e}_0, \tilde{s}_0)$ and hence

$$K_I((e', s'), x^{(e',s')})(\tilde{e}_0, \tilde{s}_0) = \bot_{(\tilde{e}_0, \tilde{s}_0)} = K_I((e_0, s_0), x)(\tilde{e}_0, \tilde{s}_0).$$

(iii) Assume $(\tilde{e}_0, \tilde{s}_0) \sqsubseteq (e', s')$ and let $x \in I(e_0, s_0)$. If $\tau(\tilde{e}_0, \tilde{s}_0) = \bot$ then $K_I((e_0, s_0), x)(e', s')_{(\tilde{e}_0, \tilde{s}_0)} = \bot_{(\tilde{e}_0, \tilde{s}_0)} = K_I((e_0, s_0), x)(\tilde{e}_0, \tilde{s}_0)$. If $\tau(\tilde{e}_0, \tilde{s}_0) \neq \bot$ then $\tau_0(e', s') = \tau_0(\tilde{e}_0, \tilde{s}_0) \neq \tau_0(e_0, s_0)$ and hence

$$K_I((e_0, s_0), x)(e', s')_{(\tilde{e}_0, \tilde{s}_0)} = \bot_{I(\tilde{e}_0, \tilde{s}_0)} = K_I((e_0, s_0), x)(\tilde{e}_0, \tilde{s}_0).$$

*Verification of $K_S$.*

Since we have proved already that $K_I$ is an $(e_0, s_0), (\tilde{e}_0, \tilde{s}_0)$–transporter for all $s_0 \in S(e)_0$ and $\tilde{s}_0 \in S(\tilde{e})_0$ it is now straightforward to prove that $K_S$ is an $e_0, \tilde{e}_0$–transporter. We omit the tedious proof.

*3. $\mathcal{U}_*$ is uniformly dense and total.*

Since $\vec{\Phi}, \vec{\Psi}$ are perfect we have dense and co–dense totalities $\vec{\Phi}_*, \vec{\Psi}_*$. For a totality $(E_*, A_*, B_*)$ on $(E, A, B)$ (i.e. $E_* \subseteq E$, $A_* \subseteq_{E_*} B$, $B_* \subseteq_{\Sigma_*(E_*,B_*)} B$) we have

$$\mathcal{U}_*[\vec{\Phi}_*](A_*, B_*) = (S_{\rm wf}, I_{\rm tot}),$$

where $(E_*, S_{\rm wf}, I_{\rm tot})$ is the least totality on $(E, S, I)$ such that for all $e \in E_*$ and $s \in S(e)$ (compare with definition 10)

If $\tau(e, s) = (\beta, a)$, then $s \in S_{\rm wf}(e)$ iff $a \in A_*(e)$, and then

$$I_{\rm tot}(e, s) \simeq B_*(a).$$

If $\tau(e, s) = (\varphi_i, s_1, f)$, then $s \in S_{\rm wf}(e)$ iff $s_1 \in S_{\rm wf}(e)$ and $f \in [I_{\rm tot}(e, s_1)$
$\to_* S_{\rm wf}]$, and then

$$I_{\rm tot}(e, s) \simeq \Phi_{*i}(I_{\rm tot}(e, s_1), \lambda x.I_{\rm tot}(e, f(x))).$$

According to definition 22 we assume that $(E_*, A_*, B_*)$ is uniformly dense and co–dense at $e_0 \in E_0$ and show that $(E_*, S_{\rm wf})$ as well as $(\Sigma_*(E_*, S_{\rm wf}), I_{\rm tot})$ are uniformly dense and co–dense at $e_0$.

Since we have already shown that $\mathcal{U}_*$ is uniformly nonempty we can define $\text{sel}_I \in \Pi_*(E_*, S_{\rm wf})$ by

$$\text{sel}_I(e, s) := \text{sel}^1_{\mathcal{U}}(A(e), \lambda a.B(e, a)).$$

*Density and co–density for $I_{\rm tot}$.*

We have to show that $I_{\rm tot} \subseteq_{\Sigma_*(E_*,S_{\rm wf})} I$ is uniformly dense and co–dense at $(e_0, s_0)$ for all $s_0 \in S(e_0)_0$. We proceed by induction on $\text{rk}(e_0, s_0)$.

*Case* $\tau(e_0, s_0) = (\beta, a_0)$. Then $I(e_0, s_0) = B(e_0, a)$ and $I_{\rm tot}(e_0, s_0) = B_*(e_0, a_0)$.

In order to show that $I_{\rm tot}$ is dense at $(e_0, s_0)$ let $x_0 \in I(e_0, s_0)_0 = B(e_0, a_0)_0$. Since $B_* \subseteq_{\Sigma_*(E_*,A_*)} B$ is uniformly dense at $(e_0, s_0)$ there is $\tilde{d} \in \Pi(\Sigma_*(E_*, A_*), B_*)$ such that $K_B((e_0, a_0), x_0) \sqsubseteq \tilde{d}$. Define $d \in \Pi(\Sigma(E, S), I)$ by

$$d(e,s) := \begin{cases} \tilde{d}(e,a) & \text{if } \tau(e,s) = (\beta, a), \\ \mathrm{sel}_I(e,s) & \text{if } \tau_0(e,s) \notin \{\beta, \bot\}, \\ \bot_{I(e,s)} & \text{if } \tau(e,s) = \bot \end{cases}$$

Clearly $d \in \Pi((\Sigma_*(E_*, S_{\mathrm{wf}}), I_{\mathrm{tot}})$ and $K_I((e_0, s_0), x_0)(e,s) \sqsubseteq d(e,s)$ for all $(e,s) \in \Sigma(E,S)$.

In order to show that $I_{\mathrm{tot}}$ is co–dense at $(e_0, s_0)$ let $x_1, \dots, x_k \in B(e_0, a_0)_0$ be inconsistent. Since the totality $B_*$ is co–dense, there are tests $\tilde{t}_1, \dots, \tilde{t}_k \in [\Sigma_*(\Sigma_*(E_*, A_*), B_*) \to_* \mathbb{B}]$ separating $x_1, \dots, x_k$. We define $t_1, \dots, t_k \in [\Sigma(\Sigma(E,S), I) \to \mathbb{B}^{\bot}]$ as follows. If $x_i = \bot_{B(e_0, a_0)}$ then $t_i((e,s),x) := \#\mathrm{t}$ for all $((e,s),x) \in \Sigma(\Sigma(E,S), I)$. Otherwise let

$$t_i((e,s),x) := \begin{cases} \tilde{t}_i((e,a),x) & \text{if } \tau(e,s) = (\beta, a), \\ \#\mathrm{f} & \text{if } \tau_0(e,s) \notin \{\beta, \bot\}, \\ \bot & \text{if } \tau(e,s) = \bot \end{cases}$$

Clearly $t_i \in [\Sigma_*(\Sigma_*(E_*, S_{\mathrm{wf}}), I_{\mathrm{tot}}) \to_* \mathbb{B}]$. Let us verify that the $t_1, \dots, t_k$ separate the $x_1, \dots, x_k$.

(s1) Assume $x_i \sqsubseteq K_I((e,s),x)((e_0,s_0))$. If $x_i = \bot_{I_{e_0,s_0}}$ then $t_i((e,s),x) = \#\mathrm{t}$. If $x_i \neq \bot_{I_{s_0}}$ then $K_I((e,s),x)(e_0,s_0) \neq \bot_{I(e_0,s_0)}$, too. Hence $\tau(e,s) = (\beta, a)$ and $x_i \sqsubseteq K_I((e,s),x)(e_0,s_0) = K_B((e,a),x)(e_0,a_0)$ It follows that

$$t_i((e,s),x) = \tilde{t}_i((e,a),x) = \#\mathrm{t}.$$

(s2) Assume $((e,s),x) \in \bigcap_{i=1}^k t_i^{-1}[\#\mathrm{t}]$. Since $\{x_1, \dots, x_k\}$ is inconsistent there must be some $i_0$ such that $x_{i_0} \neq \bot$. Since $t_{i_0}((e,s),x) = \#\mathrm{t}$ it follows from the definition of $t_{i_0}$ that $\tau(e,s) = (\beta, a)$. Hence for all $i$ such that $x_i \neq \bot$ we have $\tilde{t}_i((e,a),x) = t_i((e,s),x) = \#\mathrm{t}$. If $x_i = \bot$ then $\tilde{t}_i((e,a),x) = \#\mathrm{t}$ anyway. Therefore $((e,a),x) \in \bigcap_{i=1}^k \tilde{t}_i^{-1}[\#\mathrm{t}]$.

*Case* $\tau_0(e_0, s_0) = \varphi_i$. Then $I(e_0, s_0) = \Phi_i(I \circ \delta, I \circ p)(e_0, s_0)$. Let $L := \{(e,s) \in \Sigma_*(E_*, S_{\mathrm{wf}}) : \tau_0(e,s) = \varphi_i\}$. Note that $\delta \in [L \to_* S_{\mathrm{wf}}]$ and $p \in [\Sigma_*(L, I_{\mathrm{tot}} \circ \delta) \to_* S_{\mathrm{wf}}]$. Since $\mathrm{rk}(\delta(e_0, s_0)) < \mathrm{rk}(e_0, s_0)$, by induction hypothesis, $I_{\mathrm{tot}} \subseteq_{\Sigma_*(E_*, S_{\mathrm{wf}})} I$ is dense and co–dense at $\delta(e_0, s_0)$. Hence, by lemma 11, $I_{\mathrm{tot}} \circ \delta \subseteq_L I \circ \delta$ is dense and co–dense at $(e_0, s_0)$. Furthermore $\mathrm{rk}(p(s_0, x_0)) < \mathrm{rk}(e_0, s_0)$ for all $x_0 \in I(\delta(e_0, s_0))_0$. Hence, by induction hypothesis, $I_{\mathrm{tot}} \subseteq_{\Sigma_*(E_*, S_{\mathrm{wf}})} I$ is dense and co–dense at $p(s_0, x_0)$ and by lemma 11, $I_{\mathrm{tot}} \circ p \subseteq_{\Sigma_*(L, I_{\mathrm{tot}})} I \circ p$ is dense and co–dense at $(s_0, x_0)$ for all $x_0 \in I(\delta(e_0, s_0))_0$. Hence, since $(\Phi_i, \Phi_{*i})$ is perfect, $\Phi_{*i}(I_{\mathrm{tot}} \circ \delta, I_{\mathrm{tot}} \circ p) \subseteq_L \Phi_{*i}(I \circ \delta, I \circ p)$ is dense and co–dense at $(e_0, s_0)$.

To show that $I_{\mathrm{tot}} \subseteq_{\Sigma_*(E_*, S_{\mathrm{wf}})} I$ is dense at $(e_0, s_0)$, let $x_0 \in I(e_0, s_0)_0$. Since $\Phi_{*i}(I_{\mathrm{tot}} \circ \delta, I_{\mathrm{tot}} \circ p) \subseteq_L \Phi_i(I \circ \delta, I \circ p)$ is dense at $(e_0, s_0)$, there is $\tilde{d} \in \Pi_*(L, \Phi_{*i}(I_{\mathrm{tot}} \circ \delta, I_{\mathrm{tot}} \circ p))$ so that $\Gamma_{\Phi_i}(\Delta_\delta(K_I), \Delta_p(K_I))(s_0, x_0) \sqsubseteq \tilde{d}$. Define $d \in \Pi(\Sigma(E, S), I)$ by

$$d(e,s) := \begin{cases} \tilde{d}(e,s) & \text{if } \tau_0(e,s) = \varphi_i, \\ \mathrm{sel}_I(e,s) & \text{if } \tau_0(e,s) \notin \{\varphi_i, \bot\}, \\ \bot_{I(e,s)} & \text{if } \tau_0(e,s) = \bot \end{cases}$$

This is well defined, since if $\tau_0(e,s) = \varphi_i$ then $I(e,s) = \Phi(I \circ \delta, I \circ p)$.

Clearly $d \in \Phi_{*i}(\Sigma_*(E_*, S_{\mathrm{wf}}), I_{\mathrm{tot}})$. Furthermore, if $\tau_0(e,s) = \varphi_i$ then, by definition of $K_I$,

$$\begin{aligned} K_I((e_0, s_0), x_0)(e,s) &= \Gamma_{\Phi_i}(\Delta\delta(K_I), \Delta_p(K_I))((e_0, s_0), x_0)(e,s) \\ &\sqsubseteq \tilde{d}(e,s) \\ &= d(e,s). \end{aligned}$$

If $\tau_0(e,s) \neq \varphi_i$ then $K_I((e_0, s_0), x_0)(e,s) = \bot_{I(e,s)} \sqsubseteq d(e,s)$.

To show that $I_{\mathrm{tot}} \subseteq_{\Sigma_*(E_*, S_{\mathrm{wf}})} I$ is co–dense at $(e_0, s_0)$ let $x_1, \ldots, x_k \in I(e_0, s_0)_0$ be inconsistent. Since $\Phi_{*i}(I_{\mathrm{tot}} \circ \delta, I_{\mathrm{tot}} \circ p) \subseteq_L \Phi_i(I \circ \delta, I \circ p)$ is co–dense at $(e_0, s_0)$, there are $\tilde{t}_1, \ldots, \tilde{d}_k \in [\Sigma_*(L, \Phi_{*i}(I_{\mathrm{tot}} \circ \delta, I_{\mathrm{tot}} \circ p)) \to_* \mathbb{B}]$ separating $x_1, \ldots, x_k$. Define $\tilde{t}_1, \ldots, \tilde{d}_k \in [\Sigma(\Sigma(E, S), I) \to \mathbb{B}^\perp]$ as follows. If $x_i = \bot_{I_{(e_0, s_0)}}$ then $t_i((e,s), x) := \#\mathrm{t}$ for all $((e,s), x) \in \Sigma(\Sigma(E, S), I)$. Otherwise let

$$t_i((e,s),x) := \begin{cases} \tilde{t}_i((e,s),x) & \text{if } \tau_0(e,s) = \varphi_i, \\ \#\mathrm{f} & \text{if } \tau_0(e,s) \notin \{\varphi_i, \bot\}, \\ \bot & \text{if } \tau_0(e,s) = \bot \end{cases}$$

Clearly $t_i \in [\Sigma_*(S_{\mathrm{wf}}, I_{\mathrm{tot}}) \to_* \mathbb{B}]$. We show that $t_1, \ldots, t_k$ separate $x_1, \ldots, x_k$.

(s1) Assume $x_i \sqsubseteq K_I((e,s), x)(e_0, s_0)$. If $x_i = \bot_{I_{(e_0, s_0)}}$ then $t_i((e,s), x) = \#\mathrm{t}$. If $x_i \neq \bot_{I_{(e_0, s_0)}}$ then $K_I((e,s), x)(e_0, s_0) \neq \bot_{I_{(e_0, s_0)}}$, too. Hence $\tau_0(e,s) = \varphi_i$ and

$$x_i \sqsubseteq K_I((e,s),x)(e_0,s_0) = \Gamma_{\Phi_i}(\Delta_\delta(K_I), \Delta_p(K_I))((e,s),x)(e_0,s_0).$$

It follows that $t_i((e,s), x) = \tilde{t}_i((e,s), x) = \#\mathrm{t}$.

(s2) Assume $((e,s), x) \in \bigcap_{i=1}^k t_i^{-1}[\#\mathrm{t}]$. Since $\{x_1, \ldots, x_k\}$ is inconsistent there must be some $i_0$ such that $x_{i_0} \neq \bot$. Since $t_{i_0}((e,s), x) = \#\mathrm{t}$

it follows from the definition of $t_{i_0}$ that $\tau_0(e,s) = \varphi_i$. Hence for all $i$ such that $x_i \neq \bot_{I(e_0,s_0)}$

$$\tilde{t}_i((e,s),x) = t_i((e,s),x) = \#\mathrm{t}.$$

If $x_i = \bot_{I(e_0,s_0)}$ then $x_i \sqsubseteq \Gamma_{\Phi_i}(\Delta_\delta(K_I),\Delta_p(K_I))((e,s),x)(e_0,s_0)$. Hence $\tilde{t}_i((e,s),x) = \#\mathrm{t}$ and $x \in \bigcap_{i=1}^{k} \tilde{t}_i^{-1}[\#\mathrm{t}]$.

*Case* $\tau_0(e_0,s_0) = \bot$. Then $I(e_0,s_0) = \{\bot\}$. In order to show that $I_{\mathrm{tot}} \subseteq_{\Sigma_*(E_*,S_{\mathrm{wf}})} I$ is dense at $(e_0,s_0)$ let $x_0 \in I(e_0,s_0)_0$, i.e. $x_0 = \bot_{I(e_0,s_0)}$. Define

$$d := \mathrm{sel}_I \in \Pi_*(\Sigma_*(E_*,S_{\mathrm{wf}}),I_{\mathrm{tot}}).$$

We have

$$K_I((e_0,s_0),\bot_{I(e_0,s_0)})(e,s) = \bot_{I(e,s)} \sqsubseteq \mathrm{sel}_I(e,s) = d(e,s).$$

Trivially $I_{\mathrm{tot}}$ is co–dense at $(e_0,s_0)$ since there are no inconsistent elements in $I(e_0,s_0)$.

*Density and co–density for* $(E_*,S_{\mathrm{wf}})$.

This is shown similarly. We omit the proof.

From theorem 3 we immediately get

*Corollary.* For every $n$ the wellfounded universe operator $\mathcal{U}^n_{\mathrm{wf}}$ is perfect. Consequently the wellfounded universes $(S^{(n)}_{\mathrm{wf}},I^{(n)}_{\mathrm{tot}})$ as well as the wellfounded super universes $(S^n_{\mathrm{wf}},I^n_{\mathrm{tot}})$ are uniformly dense and codense. In particular the set $S^n_{\mathrm{wf}}$ is dense and codense in $S^n$ and for every $s \in S^n_{\mathrm{wf}}$ the set $I^n_{\mathrm{tot}}(s)$ is dense and codense in $I(s)$; similarly for $(S^{(n)}_{\mathrm{wf}},I^{(n)}_{\mathrm{tot}})$.

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