

MODIFIED BAR RECURSION AND CLASSICAL DEPENDENT CHOICE

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Abstract. We introduce a variant of Spector's bar recursion in finite types to give a realizability interpretation of the classical axiom of dependent choice allowing for the extraction of witnesses from proofs of Σ_1 formulas in classical analysis. We also give a bar recursive definition of the fan functional and study the relationship of our variant of bar recursion with others.

§1. Introduction. In [21] Spector extended Gödel's Dialectica Interpretation of Peano Arithmetic [9] to classical analysis using bar recursion in finite types. Although considered questionable from an intuitionistic point of view ([1], 6.6) there has been considerable interest in bar recursion, and several variants of this definition scheme and their interrelations have been studied by, e.g. Schwichtenberg [18], Bezem [7] and Kohlenbach [13]. In this paper we add another variant of bar recursion and use it to give a realizability interpretation of the classical, i.e. negatively translated, axiom of dependent choice that can be used to extract witnesses from proofs of Σ_1 -formulas in full classical analysis. Our interpretation is inspired by a paper by Berardi, Bezem and Coquand [2] who use a similar kind of recursion in order to interpret dependent choice. The main difference to our paper is that in [2] a rather ad-hoc infinitary term calculus and a non-standard notion of realizability are used whereas we work with a straightforward combination of negative translation, A-translation, modified realizability, and Plotkin's adequacy result for the partial continuous functional semantics of PCF [17].

As a second application of bar recursion we show that the definition of the fan functional within PCF given in [3] and [16] can be derived from Kohlenbach's and our variant of bar recursion. Furthermore, we investigate how our version of bar recursion behaves in the model of majorizable functions in an attempt to establish a relation with Spector's original definition.

§2. Bar recursion in finite types. We work in a suitable extension of Heyting Arithmetic in finite types, HA^ω . For convenience we enrich the type system by the formation of finite sequences (instead of using the usual encodings). So our *Types* are \mathbb{N} , $\rho \times \sigma$, $\rho \rightarrow \sigma$, and, in addition, ρ^* (the type of finite sequences of objects of type ρ). We also set $\rho^\omega := \mathbb{N} \rightarrow \rho$ (the type of infinite sequences of objects of type ρ). The *level* of a type is defined as usual: $\text{level}(\mathbb{N}) =$

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Basic Research in Computer Science, funded by the Danish National Research Foundation.

0, $\text{level}(\rho \times \sigma) = \max(\text{level}(\rho), \text{level}(\sigma))$, $\text{level}(\rho^*) = \text{level}(\rho)$, $\text{level}(\rho \rightarrow \sigma) = \max(\text{level}(\rho) + 1, \text{level}(\sigma))$. By o we will denote an arbitrary but fixed type of level 0, and by ρ, τ, σ arbitrary but again fixed types. Note that (in any reasonable model) the elements of type o are finite objects. We use the variables $i, j, k, l, m, n: \mathbb{N}$; $x, y: \rho$; $s, t: \rho^*$; $\alpha, \beta: \rho^\omega$. Other letters will be used for variables whose types can be inferred from the context. The same letter may have different types in different contexts. We also use the following operations on finite and infinite sequences:

$$\begin{aligned}
\langle x_0, \dots, x_{n-1} \rangle &::= \text{the finite sequence with elements } x_0, \dots, x_{n-1} \\
|s| &::= \text{the length of } s, \text{ i.e. } |\langle x_0, \dots, x_{n-1} \rangle| = n \\
s_k &::= \text{the } k\text{-th element of } s \text{ provided } k < |s| \\
s * t &::= \text{the concatenation of } s \text{ and } t \\
s * x &::= s * \langle x \rangle \\
s * \alpha &::= \text{appending } \alpha \text{ to } s, \text{ i.e.} \\
&\quad (s * \alpha)(k) = [\text{if } k < |s| \text{ then } s_k \text{ else } \alpha(k - |s|)] \\
s @ \alpha &::= \text{overwriting } \alpha \text{ with } s, \text{ i.e.} \\
&\quad (s @ \alpha)(k) = [\text{if } k < |s| \text{ then } s_k \text{ else } \alpha(k)] \\
s @ x &::= s @ \lambda k.x, \text{ i.e.} \\
&\quad (s @ x)(k) = [\text{if } k < |s| \text{ then } s_k \text{ else } x] \\
\bar{\alpha}k &::= \langle \alpha(0), \dots, \alpha(k-1) \rangle \\
\beta \in \bar{\alpha}k &::= \bar{\beta}k = \bar{\alpha}k.
\end{aligned}$$

DEFINITION 2.1. *Spector's definition of bar recursion [21] reads in our notation as follows:*

$$(1) \quad \Phi(s) \stackrel{\tau}{=} \begin{cases} G(s) & \text{if } Y(s @ 0^\rho) \stackrel{\mathbb{N}}{<} |s| \\ H(s, \lambda x. \Phi(s * x)) & \text{otherwise.} \end{cases}$$

In his thesis [13] Kohlenbach introduced the following kind of bar recursion which differs from Spector's only in the stopping condition:

$$(2) \quad \Phi(s) \stackrel{\tau}{=} \begin{cases} G(s) & \text{if } Y(s @ 0^\rho) \stackrel{o}{=} Y(s @ 1^\rho) \\ H(s, \lambda x. \Phi(s * x)) & \text{otherwise.} \end{cases}$$

Finally, we define Modified bar recursion at type ρ :

$$(3) \quad \Phi(s) \stackrel{o}{=} Y(s @ H(s, \lambda x. \Phi(s * x))).$$

Instead of $\Phi(s)$ we should have written more precisely $\Phi(Y, G, H, s)$ in (1), (2), and $\Phi(Y, H, s)$ in (3) in order to make clear that these equations specify functionals Φ of the respective types

$$\begin{aligned}
&(\rho^\omega \rightarrow \mathbb{N}) \rightarrow (\rho^* \rightarrow \tau) \rightarrow (\rho^* \rightarrow (\rho \rightarrow \tau) \rightarrow \tau) \rightarrow \rho^* \rightarrow \tau \\
&(\rho^\omega \rightarrow o) \rightarrow (\rho^* \rightarrow \tau) \rightarrow (\rho^* \rightarrow (\rho \rightarrow \tau) \rightarrow \tau) \rightarrow \rho^* \rightarrow \tau \\
&(\rho^\omega \rightarrow o) \rightarrow (\rho^* \rightarrow (\rho \rightarrow o) \rightarrow \rho^\omega) \rightarrow \rho^* \rightarrow o
\end{aligned}$$

We say a model \mathcal{S} satisfies one of the respective variants of bar recursion if in \mathcal{S} a functional exists satisfying the corresponding equation (1), (2), or (3) for all possible values of Y, G, H, s .

Recursive definitions similar to (3) occur in [2], and, in a slightly different form in [3] and [16] in connection with the fan functional (cf. section 4).

REMARK 2.2. Note that replacing in equation (3) the operation $@$ by $*$ would be an inessential change. However it *is* essential that the type of $\Phi(s)$ is of level 0. If, for example, the type of $\Phi(s)$ were $\mathbb{N} \rightarrow \mathbb{N}$ we could set $Y(\alpha)(m) \stackrel{\circ}{=} \alpha(m) + 1$ and $H(s, \Gamma)(k) \stackrel{\circ}{=} \Gamma(0)(|s|)$, and obtain the equation

$$\Phi(s)(m) \stackrel{\circ}{=} (s @ \lambda k. \Phi(s * \langle 0 \rangle)(|s|))(m) + 1$$

implying

$$\Phi(\langle \rangle)(0) = \Phi(\langle 0 \rangle)(1) + 1 = \Phi(\langle 0, 0 \rangle)(2) + 2 = \dots$$

which is inconsistent with HA^ω .

The structures of primary interest to interpret bar recursion are the model \mathcal{C} of *total continuous functionals* of Kleene [12] and Kreisel [14], the model $\widehat{\mathcal{C}}$ of *partial continuous functionals* of Scott [19] and Ershov [8], and the model \mathcal{M} of (strongly) *majorizable functionals* introduced by Howard [10] and Bezem [6].

THEOREM 2.3. *The models \mathcal{C} and $\widehat{\mathcal{C}}$ satisfy all three variants of bar recursion.*

PROOF. In the model $\widehat{\mathcal{C}}$ all three forms of bar recursion can simply be defined as the least fixed points of suitable continuous functionals. For \mathcal{C} we use Ershov's result in [8] according to which the model \mathcal{C} can be identified with the total elements of $\widehat{\mathcal{C}}$. Therefore it suffices to show that all three versions of bar recursion are total in $\widehat{\mathcal{C}}$. For Spector's version this has been shown by Ershov [8], and for the other versions similar argument apply. For example, in order to see that $\Phi(s)$ defined recursively by equation (3) is total for given total Y , H and s one uses bar induction on the bar

$$P(s): \Leftrightarrow Y(s @ \perp_\rho) \text{ is total}$$

where \perp_ρ denotes the undefined element of type ρ . $P(s)$ is a bar because Y is continuous. ⊥

THEOREM 2.4. *\mathcal{M} satisfies Spector's bar recursion, but not Kohlenbach's.*

PROOF. [13]. ⊥

The question whether \mathcal{M} satisfies modified bar recursion will be discussed in section 5.

§3. Using bar recursion to realize classical dependent choice. The aim of this section is to show how modified bar recursion can be used to extract witnesses from proofs of Σ_1^0 -formulas in classical arithmetic plus the axiom (scheme) of dependent choice [11]

$$\text{DC} \quad \forall n \forall x \exists y A(n, x, y) \rightarrow \forall x \exists f (f(0) = x \wedge \forall n A(n, f(n), f(n+1))).$$

Actually we will need only the following *weak modified bar recursion* which is the special case of equation (3) where H is constant:

$$(4) \quad \Phi(s) \stackrel{\circ}{=} Y(s @ \lambda k. H(s, \lambda x. \Phi(s * x))).$$

Note that in (4) the returning type of H is ρ , i.e., the argument of Y consists of s followed by an infinite sequence with constant value of type ρ .

Before dealing with dependent choice we discuss our extraction method in general and then give a realizer for the (simpler) classical axiom of countable choice.

3.1. Witnesses from classical proofs. The method we use to extract witnesses from classical proofs is a combination of Gödel's negative translation (translation P° in [15] page 42, see also [24]), the Dragalin/Friedman/Leivant trick, also called A-translation [23], and Kreisel's (formalized) modified realizability [22]. The method works in general for proofs in classical arithmetic in finite types, PA^ω (for simplicity without extensionality, for the problem of eliminating extensionality see [15]). In order to extend it to PA^ω plus extra axioms Γ (e.g. $\Gamma \equiv \text{DC}$) one has to find realizers for the negative translation, Γ^N , of Γ , where \perp is replaced by a Σ_1^0 -formula (regarding negation, $\neg C$, as defined by $C \rightarrow \perp$). For technical reasons we follow [4] and combine the Dragalin/Friedman/Leivant trick and modified realizability: instead of replacing \perp by a Σ_1^0 -formula we slightly change the definition of modified realizability by regarding $y \mathbf{mr} \perp$ as an (uninterpreted) atomic formula. More formally we define

$$y \mathbf{mr} \perp := P_\perp(y)$$

where P_\perp is a new unary predicate symbol. In the following proposition Δ is supposed to be an axiom system that is closed under substitution of quantifier free formulas for P_\perp , i.e. whenever $D \in \Delta$ and B is quantifier free then also $D[\lambda y^\circ . B/P_\perp]$ is in Δ , where $D[\lambda y^\circ . B/P_\perp]$ is obtained from B by replacing any occurrence of a formula $P_\perp(L)$ in B by $B[L/y]$.

PROPOSITION 3.1. *Assume that Φ is a closed term such that*

$$\text{HA}^\omega + \Delta \vdash \Phi \mathbf{mr} \Gamma$$

Then from any proof of

$$\text{PA}^\omega + \Gamma \vdash \forall z^\sigma \exists y^\circ B(z, y)$$

where $B(z, y)$ is quantifier free and does not contain P_\perp , one can extract a closed term $M^{\sigma \rightarrow \circ}$ such that

$$\text{HA}^\omega + \Delta \vdash \forall z B(z, Mz).$$

PROOF. The proof is folklore. The main steps are as follows. Assuming w.l.o.g. that $B(z, y)$ is atomic (P_\perp does not occur in $B(z, y)$) we obtain from $\text{PA}^\omega + \Gamma \vdash \forall z^\sigma \exists y^\circ B(z, y)$ via negative translation

$$\text{HA}^\omega + \Gamma^N \vdash_m \forall y (B(z, y) \rightarrow \perp) \rightarrow \perp$$

where \vdash_m denotes derivability in minimal logic, i.e. ex-falso-quodlibet is not used. Now, soundness of modified realizability (which holds for our abstract version of modified realizability and minimal logic [4]), together with the assumption on Φ allows us to extract from this proof a closed term M such that

$$\text{HA}^\omega + \Delta \vdash Mz \mathbf{mr} (\forall y (B(z, y) \rightarrow \perp) \rightarrow \perp)$$

i.e.

$$\text{HA}^\omega + \Delta \vdash \forall f^{\circ \rightarrow \circ} (\forall y (B(z, y) \rightarrow P_\perp(fy)) \rightarrow P_\perp(Mzf)).$$

Replacing P_\perp by $\lambda y . B(z, y)$ (remember the closure property of Δ) we conclude

$$\text{HA}^\omega + \Delta \vdash \forall z B(z, Mz(\lambda y . y)).$$

□

We will apply this proposition with the axiom system Δ consisting of the defining equation (3) for modified bar recursion together with the axiom of

$$\mathbf{Continuity} \quad \forall F \rho^\omega \rightarrow^o, \alpha \exists n \forall \beta (\forall i < n \alpha(i) = \beta(i) \rightarrow F(\alpha) = F(\beta))$$

(we call any n such that $\forall \beta (\forall i < n \alpha(i) = \beta(i) \rightarrow F(\alpha) = F(\beta))$ a point of continuity of F at α), and the following schema of

Relativized quantifier free bar induction

$$\forall \alpha \in S \exists n P(\bar{\alpha}n) \wedge \forall s \in S (\forall x [S(s*x) \rightarrow P(s*x)] \rightarrow P(s)) \wedge S(\langle \rangle) \rightarrow P(\langle \rangle)$$

where $S(s)$ is arbitrary, $P(s)$ is quantifier free, and $\alpha \in S$, $s \in S$ are shorthands for $\forall n S(\bar{\alpha}n)$ and $S(s)$ respectively. Clearly the condition on Δ in proposition 3.1 is satisfied. This principle of bar induction is similar to Luckhardt's general bar induction over species for quantifier free formulas $(\mathbf{aBI})_{\mathbb{D}}^{\rho}$ ([15], page 144).

In order to make sure that realizers can indeed be used to compute witnesses one needs to know that, 1. the axioms of $\mathbf{HA}^\omega + \Delta$ hold in a suitable model – we can choose the model \mathcal{C} of continuous functionals – and, 2. every closed term of type level 0 (e.g. of type \mathbb{N}) can be reduced to a numeral in an effective and provably correct way. In [2] this is solved by building the notion of reducibility to normal form into the definition of realizability. In our case we solve this problem by applying Plotkin's adequacy result [17] as follows: each term in the language of \mathbf{HA}^ω plus the bar recursive constants can be naturally viewed as a term in the language PCF [17], by defining the bar recursors by means of the general fixed point combinator. In this way our term calculus also inherits PCF's call-by-name reduction, i.e. if M is bar recursive and M reduces to M' then M' is bar recursive. Furthermore reduction is provably correct in our system, i.e. if M reduces to M' then $M = M'$ is provable. Now let M be a closed term of type \mathbb{N} . By theorem 2.3 M has a total value, which is a natural number n , in the model of partial continuous functionals. Hence, by Plotkin's adequacy theorem M reduces to the numeral denoting n .

3.2. Realizing \mathbf{AC}^N . We now construct a realizer of the classical (i.e. negatively translated) axiom of countable choice

$$\mathbf{AC} \quad \forall n \exists y A(n, y) \rightarrow \exists f \forall n A(n, f(n)).$$

The realizer for \mathbf{AC}^N is similar to the one for \mathbf{DC}^N , but technically simpler, so that the essential idea underlying the construction is more visible. Moreover we only need the following special case of relativized quantifier free bar induction:

Relativized quantifier free pointwise bar induction

$$\forall \alpha \in S \exists n P(\bar{\alpha}n) \wedge \forall s \in S (\forall x [S(x, |s|) \rightarrow P(s * x)] \rightarrow P(s)) \rightarrow P(\langle \rangle)$$

where $S(x, n)$ is arbitrary, $P(s)$ is quantifier free, and $\alpha \in S$, $s \in S$ are shorthands for $\forall n S(\alpha(n), n)$ and $\forall i < |s| S(s_i, i)$, respectively. This is similar to Luckhardt's higher bar induction over species, $(\mathbf{hBI})_{\mathbb{D}}^{\rho}$ ([15], page 144).

Note that the negative translation of \mathbf{AC} is

$$\mathbf{AC}^N \quad \forall n (\forall y (A(n, y)^N \rightarrow \perp) \rightarrow \perp) \rightarrow \forall f (\forall n A(n, f(n))^N \rightarrow \perp) \rightarrow \perp.$$

Following Spector [21] we reduce \mathbf{AC}^N to the double negation shift

$$\mathbf{DNS} \quad \forall n ((B(n) \rightarrow \perp) \rightarrow \perp) \rightarrow (\forall n B(n) \rightarrow \perp) \rightarrow \perp$$

observing that $\mathbf{AC} + \mathbf{DNS} \vdash_m \mathbf{AC}^N$, where \mathbf{DNS} is used with the formula $B(n) := \exists y A(n, y)^N$. Therefore it suffices to show that this instance of \mathbf{DNS} is

realizable. The following lemma, whose proof is trivial, is necessary to see that we can get away with the weak form (4) of modified bar recursion.

LEMMA 3.2. *Let B be a formula such that all of its atomic subformulas occur in negated form. Then there is a closed term H such that $\forall \vec{z} H \mathbf{mr} (\perp \rightarrow B)$ is provable (in minimal logic), where \vec{z} are the free variables of B (it is important here that H is closed, i.e. does not depend on \vec{z}).*

Note that the formula $B(n) := \exists y A(n, y)^N$ to which we apply **DNS** is of the form specified in lemma 3.2.

THEOREM 3.3. *The double negation shift **DNS** for a formula $B(n)$ is realizable using the weak form (4) of modified bar recursion provided $B(n)$ is of the form specified in lemma 3.2.*

PROOF. In order to realize the formula

$$\forall n ((B(n) \rightarrow \perp) \rightarrow \perp) \rightarrow (\forall n B(n) \rightarrow \perp) \rightarrow \perp$$

we assume we are given realizers

$$\begin{aligned} Y^{\rho \rightarrow o} \mathbf{mr} (\forall n B(n) \rightarrow \perp) \\ G^{\mathbb{N} \rightarrow (\rho \rightarrow o) \rightarrow o} \mathbf{mr} \forall n ((B(n) \rightarrow \perp) \rightarrow \perp) \end{aligned}$$

and try to build a realizer for \perp . Using weak modified bar recursion (4) we define

$$\Psi(s) = Y(s @ \lambda k. H(G(|s|, \lambda x^\rho. \Psi(s * x))))$$

where $H^{\rho \rightarrow \rho}$ is a closed term such that $\forall k H \mathbf{mr} (\perp \rightarrow B(k))$ is provable, according to lemma 3.2. We set

$$\begin{aligned} S(x, n) &:= x \mathbf{mr} B(n), \\ P(s) &:= \Psi(s) \mathbf{mr} \perp, \end{aligned}$$

and, by quantifier free pointwise bar induction relativized to S , we show $P(\langle \rangle)$, i.e. $\Psi(\langle \rangle) \mathbf{mr} \perp$.

i) $\forall \alpha \in S \exists n P(\bar{\alpha}n)$. Let $\alpha \in S$ be fixed, and let n be the point of continuity of Y at α , according to the continuity axiom. By assumptions on α and Y we get $\forall \beta Y(\bar{\alpha}n @ \beta) \mathbf{mr} \perp$, which implies $\Psi(s) \mathbf{mr} \perp$.

ii) $\forall s \in S (\forall x [S(x, |s|) \rightarrow P(s * x)] \rightarrow P(s))$. Let $s \in S$ be fixed. Suppose $\forall x [S(x, |s|) \rightarrow P(s * x)]$, i.e. $\forall x [x \mathbf{mr} B(|s|) \rightarrow \Psi(s * x) \mathbf{mr} \perp]$, i.e.

$$\lambda x^\rho. \Psi(s * x) \mathbf{mr} (B(|s|) \rightarrow \perp).$$

Using the assumption on G we obtain

$$G(|s|, \lambda x^\rho. \Psi(s * x)) \mathbf{mr} \perp,$$

and from that, setting $z := H(G(|s|, \lambda x^\rho. \Psi(s * x)))$, we obtain $y \mathbf{mr} B(k)$, for all k . Because $s \in S$ it follows that $s @ z \mathbf{mr} \forall k B(k)$ and therefore

$$Y(s @ \lambda k. z) \mathbf{mr} \perp.$$

Since $\Psi(s) = Y(s @ \lambda k. z)$ we have $P(s)$. ⊣

As explained above theorem 3.3 yields

COROLLARY 3.4. *The negative translation of the countable axiom of choice, \mathbf{AC}^N is realizable using the weak form (4) of modified bar recursion.*

3.3. Realizing \mathbf{DC}^N . With a similar, but technically somewhat more involved construction we now prove

THEOREM 3.5. *The negative translation of the axiom of dependent choice, \mathbf{DC}^N , is realizable using the weak form (4) of modified bar recursion .*

PROOF. Let σ be the type of realizers of $A(n, x, y)$. Given x_0^ρ and realizers

$$\begin{aligned} G^{\mathbb{N} \rightarrow \rho \rightarrow (\rho \rightarrow \sigma \rightarrow \circ) \rightarrow \circ} \mathbf{mr} \forall n \forall x (\forall y (A(n, x, y)^N \rightarrow \perp) \rightarrow \perp) \\ Y^{\rho^\omega \rightarrow \sigma^\omega \rightarrow \circ} \mathbf{mr} \forall f (f(0) = x_0 \wedge \forall n A(n, f(n), f(n+1)) \rightarrow \perp) \end{aligned}$$

we have to construct a realizer of \perp . In the rest of this proof the variables β and t have the types $(\rho \times \sigma)^\omega$ and $(\rho \times \sigma)^*$ respectively (deviating from our general conventions). First we perform a trivial transformation on Y defining

$$\tilde{Y}^{(\rho \times \sigma)^\omega \rightarrow \circ}(\beta) := Y(x_0 * (\pi_0 \circ \beta), \pi_1 \circ \beta)$$

where π_0, π_1 are the left and right projection and \circ is composition of functions. Using weak bar recursion (4) we now define

$$\Psi(t) = \tilde{Y}(t @ \pi(0^\rho, H(G(n, (x_0 * (\pi_0 \circ t))_{|t|}, \lambda y^\rho \lambda z^\sigma . \Psi(t * \pi(y, z))))))$$

where $\forall k, x, y \mathbf{Hmr} (\perp \rightarrow A(k, x, y)^N)$ according lemma 3.2, $\pi(\cdot, \cdot)$ is pairing, and $\pi_0 \circ t := \langle \pi_0(t_0), \dots, \pi_0(t_{|t|}) \rangle$ (hence $(\pi_0 \circ t)_i = \pi_0(t_i)$ for $i < |t|$). We define predicates

$$\begin{aligned} S(t) &:= \forall i < |t| \pi_1(t_i) \mathbf{mr} A(i, (x_0 * (\pi_0 \circ t))_i, (\pi_0 \circ t)_i)^N \\ P(t) &:= \Psi(t) \mathbf{mr} \perp. \end{aligned}$$

We show $P(\langle \rangle)$ by quantifier free bar induction relativized to S . Obviously $S(\langle \rangle)$ holds.

i) $\forall \beta \in S \exists n P(\bar{\beta}n)$. Let $\beta \in S$. Set $f^{\rho^\omega} := x_0 * (\pi_0 \circ \beta)$ and $\gamma^{\sigma^\omega} := \pi_1 \circ \beta$. Then $f(0) = x_0$ and $\forall n \gamma(n) \mathbf{mr} A(n, f(n), f(n+1))$. Therefore $Y(f, \gamma) \mathbf{mr} \perp$. Let n be a point of continuity of \tilde{Y} at β . Then

$$\Psi(\bar{\beta}n) = \tilde{Y}(\beta) = Y(f, \gamma)$$

and therefore $\Psi(\bar{\beta}n) \mathbf{mr} \perp$, i.e. $P(\bar{\beta}n)$.

ii) $\forall t \in S (\forall q^{\rho \times \sigma} [S(t * q) \rightarrow P(t * q)] \rightarrow P(t))$. Let $t \in S$ where, say, $t = \langle \pi(x_1, z_0) \dots \pi(x_n, z_{n-1}) \rangle$. Assume further $\forall q [S(t * q) \rightarrow P(t * q)]$, i.e.

$$\forall x_{n+1}, z_n [\forall i \leq n z_i \mathbf{mr} A(i, x_i, x_{i+1})^N \rightarrow \Psi(\langle x_1, z_0, \dots, x_{n+1}, z_n \rangle) \mathbf{mr} \perp]$$

Because $t \in S$ it follows that

$$\forall x_{n+1}, z_n [z_n \mathbf{mr} A(n, x_n, x_{n+1})^N \rightarrow \Psi(\langle \pi(x_1, z_0), \dots, \pi(x_{n+1}, z_n) \rangle) \mathbf{mr} \perp]$$

i.e.

$$\lambda y \lambda z . \Psi(t * \pi(y, z)) \mathbf{mr} \forall y (A(n, x_n, y) \rightarrow \perp).$$

By the assumption on G it follows $G(n, x_n, \lambda y \lambda z . \Psi(t * \pi(y, z))) \mathbf{mr} \perp$ and hence for $a := H(G(n, x_n, \lambda y \lambda z . \Psi(t * \pi(y, z))))$ we have $\forall k, x, x' (a \mathbf{mr} A(k, x, x'))$. Now we set $f^{\rho^\omega} := \langle x_0, x_1, \dots, x_n \rangle @ 0^\rho$ and $\gamma^{\sigma^\omega} := \langle z_0, \dots, z_{n-1} \rangle @ a$. Then $\forall n \gamma(n) \mathbf{mr} A(n, f(n), f(n+1))$ and therefore $Y(f, \gamma) \mathbf{mr} \perp$. But, because $x_n = (x_0 * (\pi_0 \circ t))_{|t|}$ we have

$$\Psi(t) = \tilde{Y}(t @ \pi(0^\rho, a)) = Y(f, \gamma).$$

Hence $\Psi(t) \mathbf{mr} \perp$, i.e. $P(t)$. +

§4. Bar recursion and the fan functional. A functional $\text{FAN}^{(\mathbb{N}^\omega \rightarrow o) \rightarrow \mathbb{N}}$ is called *fan functional* if it computes a modulus of uniform continuity for every functional $Y^{\mathbb{N}^\omega \rightarrow o}$ restricted to infinite 0, 1-sequences, i.e. if FAN satisfies

$$\forall Y; \alpha, \beta \leq \lambda x.1. (\bar{\alpha}(\text{FAN}(Y)) = \bar{\beta}(\text{FAN}(Y)) \rightarrow Y\alpha \stackrel{o}{=} Y\beta)$$

(note that $\rho = \mathbb{N}$ here). A recursive algorithm for $\text{FAN}(Y)$ that was given in [3] and [16] uses two procedures,

$$(5) \quad \Phi(s, v) = s @ [\text{if } Y(\Phi(s * 0, v)) \neq v \text{ then } \Phi(s * 0, v) \text{ else } \Phi(s * 1, v)]$$

$$(6) \quad \Psi(s) \stackrel{\mathbb{N}}{=} \begin{cases} 0 & \text{if } Y(\alpha) = Y(s @ 0), \\ & \text{where } \alpha = \Phi(s, Y(s @ 0)) \\ 1 + \max\{\Psi(s * 0), \Psi(s * 1)\} & \text{otherwise.} \end{cases}$$

The first functional, $\Phi(s, v)$, returns an infinite path α having s as a prefix, such that $Y(s @ \alpha) \neq v$, if such a path exists, and returns s extended by $\lambda x.1$, otherwise, i.e. if Y is constant v on all paths extending s . The second functional, $\Psi(s)$, returns the maximum point of continuity for Y on all extension of s . Therefore, a fan functional can be defined as $\text{FAN}(Y) := \Psi(\langle \rangle)$.

THEOREM 4.1. *The functional FAN can be defined using bar recursion (3) and (2) together.*

Before we give the proof of the theorem we prove two lemmas.

LEMMA 4.2. *Modified bar recursion (3) is equivalent to*

$$(7) \quad \Phi(s^{\rho^*}) \stackrel{o}{=} Y(s @ H(s, \lambda t^{\rho^*} \lambda x^{\rho}. \Phi(s * t * x)))$$

and also to

$$(8) \quad \Phi(s^{\rho^*}) \stackrel{o^\omega}{=} s @ H(s, \lambda t^{\rho^*} \lambda x^{\rho}. Y^{\rho^\omega \rightarrow o}(\Phi(s * t * x))).$$

PROOF. Obviously equation (7) subsumes modified bar recursion. It is also easy to see that equations (7) and (8) are equivalent: Given Φ satisfying (7) we define $\Phi'(s) := s @ H(s, \lambda t \lambda x. \Phi(s * t * x))$ which satisfies (8), provably by relativized bar induction. Conversely, if Φ' satisfies (8) then Φ defined by $\Phi(s) := Y(\Phi'(s))$ satisfies (7). Furthermore it is clear that we can replace the operation $@$ in each of the equations (3), (7) and (8) by $*$, i.e. we prefix with s instead of overwriting (see the definitions at the beginning of section 2). Hence it suffices to show that we can define a functional Φ satisfying

$$(9) \quad \Phi(s^{\rho^*}) \stackrel{o}{=} Y(s * H(s, \lambda t^{\rho^*} \lambda x^{\rho}. \Phi(s * t * x)))$$

by modified bar recursion. To this end we will use equation (3) (where $@$ is replaced by $*$) at type ρ^* . We define *freeze*: $\rho^* \rightarrow \rho^{**}$ and *melt*: $\rho^{**} \rightarrow \rho^*$ by $\text{freeze}(\langle x_0, \dots, x_{n-1} \rangle) = \langle \langle x_0 \rangle, \dots, \langle x_{n-1} \rangle \rangle$, $\text{melt}(\langle s_0, \dots, s_{n-1} \rangle) = s_0 * \dots * s_{n-1}$, so that $\text{melt}(\text{freeze}(s)) = s$. Given $Y^{\rho^\omega \rightarrow o}$ and $H^{\rho^* \rightarrow (\rho \rightarrow o) \rightarrow \rho^\omega}$ we define using modified bar recursion (3)

$$\Psi(q) = Y(\text{melt}(q) * H(\text{melt}(q), \lambda x \lambda s. \Psi(q * (s * x)))).$$

By relativized bar recursion one easily proves

$$\forall q, q' (\text{melt}(q) = \text{melt}(q') \rightarrow \Psi(q) = \Psi(q')).$$

Using this one easily proves by relativized bar recursion that Φ , defined by $\Phi(s) := \Psi(\text{freeze}(s))$, satisfies (9). \dashv

LEMMA 4.3. *Kohlenbach's Bar recursion (2) is equivalent to,*

$$(10) \quad \Phi(s) \stackrel{\tau}{=} \begin{cases} G(s) & \text{if } Y(s @ 0^\rho) \stackrel{o}{=} Y(s @ J(s)) \\ H(s, \lambda x^\rho. \Phi(s * x)) & \text{otherwise,} \end{cases}$$

where the new parameter J is of type $\rho^* \rightarrow \rho^\omega$ and, as usual, $\Phi(s)$ is shorthand for the more accurate $\Phi(Y, G, H, J, s)$.

PROOF. Our proof is based on the proof of theorem 3.66 in [13]. The fact that (2) can be defined from (10) is trivial. To define (10) from (2) one uses the following trick. For s^{ρ^*} , $s + (\dot{-})k$ denotes pointwise addition (cut-off subtraction) of appropriate type, and $\kappa(n) := n$, $\kappa(f^{\rho \rightarrow \sigma}) := \kappa(f(0^\rho))$, $\kappa(z^{\rho \times \sigma}) := \kappa(\pi_0(z))$, so $\kappa(x^\rho + 2) > 1$ and $\kappa(n^\rho) = n$. Define

$$\eta(\beta^{\rho^\omega})(n) := \begin{cases} \beta(n) \dot{-} 2 & \text{if } \kappa(\beta(n)) > 1 \\ J(\phi(\bar{\beta}n))(n) & \text{if } \kappa(\beta(n)) = 1 \\ 0 & \text{if } \kappa(\beta(n)) = 0, \end{cases}$$

where $\phi(s) := \langle s_0, \dots, s_{(\mu k < |s|. (s_k=1)) - 1} \rangle$. Clearly

$$\begin{aligned} \eta((s+2) @ 0^\rho) &= s @ 0^\rho, \\ \eta((s+2) @ 1^\rho) &= Y(s @ J(s)). \end{aligned}$$

Now define using Kohlenbach's bar recursion (2)

$$\tilde{\Phi}(s) \stackrel{\tau}{=} \begin{cases} G(s \dot{-} 2) & \text{if } Y(\eta(s @ 0^\rho)) = Y(\eta(s @ 1^\rho)) \\ H(s, \lambda x^\rho. \tilde{\Phi}(s * x)) & \text{otherwise.} \end{cases}$$

Then clearly $\Phi(s) := \tilde{\Phi}(s+2)$ satisfies (10). \dashv

PROOF OF THEOREM 4.1. We show that procedures Φ and Ψ satisfying the equations (5) and (6) respectively can be defined using equations (3) and (2).

For defining the functional $\Phi(s, v)$ we use equation (8) of lemma 4.2.

$$\Phi(s, v) \stackrel{1}{=} s @ H(s, v, \lambda t \lambda x. Y(\Phi(s * t * x)))$$

where H is defined by course of value primitive recursion as

$$H(s, v, \Gamma)(n) \stackrel{o}{=} \begin{cases} s_n & \text{if } n < |s| \\ 0 & \text{if } n \geq |s| \wedge \Gamma(c, 0) \neq v \\ 1 & \text{if } n \geq |s| \wedge \Gamma(c, 0) = v, \end{cases}$$

with $c := \langle H(s, v, \Gamma)(|s|), \dots, H(s, v, \Gamma)(n-1) \rangle$. Clearly Φ satisfies equation (5) at all $n \leq |s|$. For $n > |s|$ we first observe that

$$\Phi(s, v)(n) \stackrel{o}{=} \begin{cases} 0 & \text{if } Y(\Phi(s * c_{s,n} * 0)) \neq v \\ 1 & \text{if } Y(\Phi(s * c_{s,n} * 0)) = v, \end{cases}$$

where $c_{s,n} := \langle \Phi(s, v)(|s|), \dots, \Phi(s, v)(n-1) \rangle$. Now if $Y(\Phi(s * 0)) \neq v$ then $\Phi(s, v)(|s|) = 0$ and therefore $s * c_{s,n} = s * 0 * c_{s * 0, n}$. Hence $\Phi(s)(n) = \Phi(s * 0)(n)$ as required by (5). The case $Y(\Phi(s * 0)) = v$ is similar.

One immediately sees that a functional Ψ satisfying (6) can be defined from an instance of equation (10) using the functional Φ above. \dashv

§5. Modified bar recursion and the model \mathcal{M} . In [13] it is shown that the scheme of bar recursion (2) is provably not primitive recursively definable from (1), since (1) yields a well defined functional in the model of (strongly) majorizable functionals \mathcal{M} (cf. [6]) and (2) does not (in the following we will by “majorizable” always mean “strongly majorizable”). Equation (1), however, can be primitive recursively defined from (2) (cf. [13]). In this section we attack the problem of interdefinability of equations (1) and (3) by investigating whether equation (3) always has a solution in the model \mathcal{M} or not. More precisely, we ask the question: Is there a functional¹

$$\bar{\Phi} : \mathcal{M}_{\rho^\omega \rightarrow o} \times \mathcal{M}_{\rho^* \times (\rho \rightarrow o) \rightarrow \rho^\omega} \times \mathcal{M}_{\rho^*} \rightarrow \mathcal{M}_o,$$

satisfying equation (3)? We show in theorem 5.7 that, if such $\bar{\Phi}$ exists then $\bar{\Phi}$ must belong to \mathcal{M} . At the end of this section we investigate if such functional indeed exists.

A first difference to the situation in the continuous functionals is that in the model of majorizable functionals, for fixed $Y, H \in \mathcal{M}$, the solutions ($\Phi : \rho^* \rightarrow o$) for equation (3) need not be unique (in fact, even for the weakest form of (3) considered here, where H does not make use of k nor s). Let $\rho = o$. If we take $H(s, \alpha, k)$ to be $\alpha(0)$, equation (3) reduces to $\Phi(s) = Y(s @ \lambda k. \Phi(s * 0))$. Taking Y to be,

$$Y(\alpha) = \begin{cases} 1 & \text{if } \exists n \forall m \geq n (\alpha(m) = 1) \\ 0 & \text{otherwise,} \end{cases}$$

we have that equation (3) has infinitely many solutions in \mathcal{M} , e.g. $\Phi = \lambda s.1$ or $\Phi = \lambda s.0$. Notice that Y and H belong to \mathcal{M} . Moreover, from the example above we see that, in \mathcal{M} , it can be the case that for all s the value of $B(s)$ depends on the values of $B(s * x)$, which does not happen if equation (1) is interpreted in \mathcal{C} .

Recall that for continuous functionals Y of type $\rho^\omega \rightarrow o$ it is the case that from some initial segment of α the value of $Y(\alpha)$ is determined. For the majorizable functions this does not hold, but a “weak continuity” property does hold. It says that a bound on the value of $Y(\alpha)$ can be determined from an initial segment of α . For the rest of this section all variables (unless stated otherwise) are assumed to range over the type structure \mathcal{M} .

LEMMA 5.1 ([6], 1.4, 1.5). *Let \max^ρ be inductively defined as,*

$$\begin{aligned} \max_{i \leq n}^0 m_i &::= \max\{m_0, \dots, m_n\}, \\ \max_{i \leq n}^{\tau \rightarrow \rho} X_i &::= \lambda Y^\rho. \max_{i \leq n}^\rho X_i Y, \end{aligned}$$

and for α^{ρ^ω} , define $\alpha^+(n) ::= \max_{i \leq n}^\rho \alpha(i)$. Then,

$$\forall n (\alpha(n) \text{ maj } \beta(n)) \rightarrow \alpha^+ \text{ maj } \beta^+, \beta.$$

We also use addition in all types, which is done pointwise, e.g. if x, y are of type $\tau \rightarrow \rho$ then $x +_{\tau \rightarrow \rho} y ::= \lambda z^\tau (x(z) +_\rho y(z))$.

LEMMA 5.2 (Weak continuity for \mathcal{M}). $\forall Y^{\rho^\omega \rightarrow \mathbb{N}}; \alpha \exists n^\mathbb{N} \forall \beta \in \bar{\alpha}n (Y(\beta) \leq n)$.

PROOF. Let Y and α be fixed, $\alpha^* \text{ maj } \alpha$ and $Y^* \text{ maj } Y$. From the assumption

¹By $\mathcal{M}_\rho \rightarrow \mathcal{M}_\tau$ we mean an arbitrary function from \mathcal{M}_ρ to \mathcal{M}_τ . By $\mathcal{M}_{\rho \rightarrow \tau}$ we mean a functional from \mathcal{M}_ρ to \mathcal{M}_τ which belongs to \mathcal{M} .

$$(a) \forall n \exists \beta \in \bar{\alpha}n(Y(\beta) > n)$$

we derive a contradiction. For any n , let β_n be the function whose existence we are assuming in (a). Let

$$\beta_n^*(i) := \begin{cases} 0^\rho & i < n \\ [\beta_n(i)]^* & i \geq n, \end{cases}$$

where $[\beta_n(i)]^*$ denotes some majorant of $\beta_n(i)$. Having defined the functional β_n^* we note two of its properties,

- i) $\forall i < n(\beta_n^*(i) = 0^\rho)$,
- ii) $(\alpha^* +_\rho \beta_n^*)^+ \text{maj } \beta_n$. (by lemma 5.1)

Consider the functional $\hat{\alpha}$ defined as, $\hat{\alpha}(n) := \alpha^*(n) +_\rho \sum_{i \in \mathbb{N}} \beta_i^*(n)$. Since at each point n only finitely many β_i^* are non-zero, α^* is well defined. Let $Y^*(\hat{\alpha}^+) = l$. Note that $\hat{\alpha}^+ \text{maj } \beta_i$, for all $i \in \mathbb{N}$, and from (a) we should have $l < Y(\beta_l) \leq l$, a contradiction. \dashv

We extend, for convenience, the definition of majorability for finite sequences. For s^* , s of type ρ^* , $s^* \text{maj } s$ is defined as,

$$\begin{aligned} |s^*| &= |s| = 0, \text{ or} \\ |s| = 0 \wedge |s^*| \neq 0 \wedge s^* \text{maj } s^*, \text{ or} \\ s^*, s &\in (\mathcal{M}_\rho)^* \wedge \forall m < |s^*|, n < |s| (m \geq n \rightarrow s^*(m) \text{maj } s(n)), \end{aligned}$$

where $(\mathcal{M}_\rho)^*$ denotes all finite tuples of elements from \mathcal{M}_ρ . Note that,

$$s^* \text{maj}^{\rho^*} s \Leftrightarrow (s^* @ \lambda i. s^*_{|s^*|-1}) \text{maj}^{\rho^\omega} (s @ \lambda i. s_{|s|-1}),$$

where $s_{|s|-1} = 0^\rho$ if $s = \langle \rangle$.

LEMMA 5.3. *Let α^{ρ^ω} and $n \in \mathbb{N}$ be fixed. Moreover, let s^{ρ^*} be such that $|s| = n$ and $\bar{\alpha}n \text{maj } s$. Then, $\forall \beta \in s \exists \beta^* \in \bar{\alpha}n (\beta^* \text{maj } \beta)$.*

PROOF. Let α, s, n , and $\beta \in s$ be fixed. Assume $\bar{\alpha}n \text{maj } s$. Define β^* as,

$$\beta^*(i) := \begin{cases} \alpha(i) & \text{if } i < n \\ \max^\rho \{ \max_{j < i}^\rho \beta^*(j), [\beta(i)]^* \} & \text{otherwise,} \end{cases}$$

where $[\beta(i)]^*$ is some majorant of $\beta(i)$. First note that, for all i , $\beta^*(i) \text{maj } \beta(i)$. Let $i \leq k$.

If $k < n$ then $\beta^*(k) = \alpha(k) \text{maj } \alpha(i) \text{maj } s_i = \beta(i)$. (Note $\alpha(i) = \beta^*(i)$.)

If $k \geq n$ then $\beta^*(k) = \max^\rho \{ \max_{j < k}^\rho \beta^*(j), [\beta(k)]^* \} \text{maj } \beta^*(i) \text{maj } \beta(i)$. \dashv

In the following we define two functionals Ω and Γ , (functional Ω is defined in [13], 3.40) and prove some of their properties.

LEMMA 5.4. *Define $\min^\rho : 2^\rho \rightarrow \rho$ and $\Omega : \rho \rightarrow \rho$ as,*

$$\begin{aligned} \min^\circ X &:= \min X, \text{ for } \emptyset \neq X \subseteq \mathbb{N}, \\ \min^{\rho \rightarrow \tau} X &:= \lambda y^\rho. \min^\tau \{ Fy : F \in X \}, \text{ for } \emptyset \neq X \subseteq \mathcal{M}_{\rho \rightarrow \tau}, \\ \Omega(F) &:= \min^\rho \{ F^* : F^* \text{maj } F \}. \end{aligned}$$

Then,

- i) For all F , $\Omega(F) \text{maj } F$,
- ii) $\Omega \text{maj } \Omega$, (therefore, $\Omega \in \mathcal{M}$)
- iii) $(\forall F \in X \cup Y \exists F^* \in X (F^* \text{maj } F)) \rightarrow (\min^\rho X) \text{maj } (\min^\rho Y)$,
- iv) $\forall n^{\mathbb{N}}; \alpha^{\rho^\omega}, \beta (\Omega(\alpha)n = \Omega(\bar{\alpha}n @ \beta)n)$,

- v) $\forall \alpha^*, \alpha, s \ (\alpha^* \text{ maj } \alpha \rightarrow \Omega(s @ \alpha^*) \text{ maj } \Omega(s @ \alpha), s @ \alpha)$,
- vi) $\forall x^\rho \ (F(x) \geq G(x)) \rightarrow \Omega(F) \text{ maj } \Omega(G), G. \ (F, G \text{ of type } \rho \rightarrow \mathbb{N})$

PROOF. i), ii) and vi) are proven in [13], lemma 3.41.

iii) By induction on types. For $\rho = o$ the result is trivial. Assume $X, Y \subseteq \mathcal{M}_{\rho \rightarrow \tau}$ and,

$$\forall F \in X \cup Y \exists F^* \in X \ (F^* \text{ maj } F).$$

We have to show, $\lambda y. \min^\tau \{Fy : F \in X\} \text{ maj } \lambda y. \min^\tau \{Fy : F \in Y\}$. Let $y^* \text{ maj } y$. By our assumption on X and Y , and the induction hypothesis, we have,

$$\min^\tau \{Fy^* : F \in X\} \text{ maj } \min^\tau \{Fy : F \in X\}, \min^\tau \{Fy : F \in Y\}.$$

- iv) Let $i < n$, α^{ρ^ω} , β be fixed. First we note that, by lemma 5.3, for $i < n$,
- $$\{\alpha^*(i) : \alpha^* \text{ maj } \alpha\} = \{\alpha^*(i) : \alpha^* \text{ maj } \bar{\alpha}n @ \beta\}.$$

By the definition of \min^{ρ^ω} , and the observation above, we have,

$$\begin{aligned} \Omega(\alpha)(i) &= \min^\rho \{\alpha^*(i) : \alpha^* \text{ maj } \alpha\} \\ &= \min^\rho \{\alpha^*(i) : \alpha^* \text{ maj } \bar{\alpha}n @ \beta\} = \Omega(\bar{\alpha}n @ \beta)(i). \end{aligned}$$

- v) Let $\alpha^* \text{ maj } \alpha$ and s be fixed. Let $X = \{\hat{\alpha} : \hat{\alpha} \text{ maj } s @ \alpha^*\}$ and $Y = \{\hat{\alpha} : \hat{\alpha} \text{ maj } s @ \alpha\}$. From lemma 5.3 we can derive, $\forall \alpha_1 \in X \cup Y \exists \alpha_2 \in X \ (\alpha_2 \text{ maj } \alpha_1)$. And by iii), $\Omega(s @ \alpha^*) = \min^\rho X \text{ maj } \min^\rho Y = \Omega(s @ \alpha)$. \dashv

LEMMA 5.5. Define $\Gamma : (\rho^\omega \rightarrow \mathbb{N}) \rightarrow (\rho^\omega \rightarrow \mathbb{N})$ as,

$$\Gamma(Y)(\alpha) := \mu n [\forall \beta \in \bar{\alpha}n (\Omega(Y)(\beta) \leq n)].$$

Then,

- i) $\Gamma(Y) \text{ maj } Y$,
- ii) $\Gamma(Y) \in \mathcal{C}$ and $n = \Gamma(Y)(\alpha)$ is a point of continuity for $\Gamma(Y)$ on α ,
- iii) $\Gamma \text{ maj } \Gamma$. (therefore, $\Gamma \in \mathcal{M}$)

PROOF. First of all, we note that, by lemma 5.2, the functional Γ is well defined. By lemma 5.4 (i), $\Omega(Y) \text{ maj } Y$.

- i) Let $\alpha^* \text{ maj } \alpha$. We have to show $\Gamma(Y)(\alpha^*) \geq \Gamma(Y)(\alpha), Y(\alpha)$. By the definition of $\Gamma(Y)$, and lemma 5.4 (ii), we have $\Gamma(Y)(\alpha^*) \geq \Omega(Y)(\alpha^*) \geq Y(\alpha)$. Suppose that $n = \Gamma(Y)(\alpha^*) < \Gamma(Y)(\alpha) = m$. Note that there exists a $\beta \in \bar{\alpha}(m-1)$ such that $\Gamma(Y)(\beta) \geq m$ (otherwise we get a contradiction to the minimality in the definition of $\Gamma(Y)$). But since $m > n$, by lemma 5.3, there exists a $\beta^* \in \bar{\alpha^*}n$ such that $\beta^* \text{ maj } \beta$. Therefore, $\Omega(Y)(\beta^*) \leq n < m \leq \Omega(Y)(\beta)$, a contradiction.

- ii) Let α be fixed and take $n = \Gamma(Y)(\alpha)$. Suppose there exists a $\beta \in \bar{\alpha}n$ such that $\Gamma(Y)(\beta) \neq n$. If $\Gamma(Y)(\beta) < n$ we get, since $\alpha \in \bar{\beta}n$, that $\Gamma(Y)(\alpha) < n$, a contradiction. Suppose $\Gamma(Y)(\beta) > n$. Since $\beta \in \bar{\alpha}n$ we have, $\forall \gamma \in \bar{\beta}n (\Omega(Y)(\gamma) \leq n)$, also a contradiction.

- iii) Assume $Y^* \text{ maj } Y$ and $\alpha^* \text{ maj } \alpha$, we show $\Gamma(Y^*)(\alpha^*) \geq \Gamma(Y)(\alpha)$. By the self majorability of $\Gamma(Y)$ we have $\Gamma(Y)(\alpha^*) \geq \Gamma(Y)(\alpha)$. We now show $\Gamma(Y^*)(\alpha^*) \geq \Gamma(Y)(\alpha^*)$. Let $n = \Gamma(Y^*)(\alpha^*)$ and suppose $m = \Gamma(Y)(\alpha^*) > n$. By the definition of $\Gamma(Y)$, there exists a $\beta \in \bar{\alpha^*}(m-1)$ s.t. $\Omega(Y)(\beta) \geq m$. But, since $m > n$, by lemma 5.3, there exists a $\beta^* \in \bar{\alpha^*}n$ s.t. $\beta^* \text{ maj } \beta$, and by lemma 5.4 (ii), $\Omega(Y^*)(\beta^*) \geq m > n$, a contradiction. \dashv

LEMMA 5.6. *Let $Y^* \text{ maj } Y$ (of type $\rho^\omega \rightarrow \mathbb{N}$), $\alpha \text{ maj } \alpha$, and $n = \Gamma(Y^*)(\alpha)$. Then,*

- i) *For all $\beta \in \bar{\alpha}n$, $n \geq \Gamma(Y)(\beta)$.*
- ii) *If $\bar{\alpha}n \text{ maj } s$ and $|s| = n$ then, for all $\beta \in s$, $n \geq Y(\beta)$.*

PROOF. i) Suppose there exists a $\beta \in \bar{\alpha}n$ such that $n < \Gamma(Y)(\beta)$. Since $\alpha \text{ maj } \alpha$, by lemma 5.3, there exists a $\beta^* \in \bar{\alpha}n$ such that $\beta^* \text{ maj } \beta$, and by the self majorability of $\Gamma(Y)$, $n < \Gamma(Y)(\beta^*)$. By the fact that n is a point of continuity for $\Gamma(Y^*)$ on α we have $\Gamma(Y^*)(\beta^*) = n$. And by lemma 5.5 (iii), we get a contradiction.

ii) Let $\bar{\alpha}n \text{ maj } s$ and $|s| = n$. Suppose, there exists a $\beta \in s$ s.t. $n < Y(\beta)$. By lemma 5.3, there exists a $\beta^* \in \bar{\alpha}n$ such that $\beta^* \text{ maj } \alpha, \beta$, and therefore, by lemma 5.5 (i) and (iii), $\Gamma(Y^*)(\beta^*) \geq \Gamma(Y)(\beta^*) > n$. And by lemma 5.5 (ii), we have a contradiction. \dashv

THEOREM 5.7. *Suppose Φ is a functional which for any given $Y, H, s \in \mathcal{M}$ (of appropriate types) satisfies equation (3), then $\Phi \in \mathcal{M}$.*

PROOF. Our proof is based on the proof of the main result of [6]. The idea is that, if Φ satisfies equation (3) then the functional

$$\Phi^* := \lambda Y, H, \Omega(\Phi \hat{Y} \hat{H}) \text{ maj } \Phi,$$

where $\hat{Y}(\alpha) := \Gamma(Y)(\Omega(\alpha))$ and $\hat{H}(s, \alpha) := H(\Omega(s), \Omega(\alpha))$. Let $Y^* \text{ maj } Y$ and $H^* \text{ maj } H$ be fixed. The fact that $\Phi^* \text{ maj } \Phi$ follows, by lemma 5.4 (vi), from $\forall s P(s)$, where

$$P(s) := \Phi(\hat{Y}^*, \hat{H}^*, s) \geq \Phi(\hat{Y}, \hat{H}, s), \Phi(Y, H, s).$$

We prove $\forall s P(s)$ by bar induction:

i) $\forall \alpha \exists n P(\bar{\alpha}n)$. Let α be fixed and $n = \hat{Y}^*(\alpha) = \Gamma(Y^*)(\Omega(\alpha))$. By lemma 5.4 (iv) and lemma 5.5 (ii), $\Phi(\hat{Y}^*, \hat{H}^*, \bar{\alpha}n) = n$, and by lemma 5.6, we have $n \geq \Phi(\hat{Y}, \hat{H}, \bar{\alpha}n)$ and $n \geq \Phi(Y, H, \bar{\alpha}n)$.

ii) $\forall s (\forall x P(s * x) \rightarrow P(s))$. Let s be fixed. Assume that for all x ,

$$\underbrace{\Phi(\hat{Y}^*, \hat{H}^*, s * x)}_{\Phi_1(x)} \geq \underbrace{\Phi(\hat{Y}, \hat{H}, s * x)}_{\Phi_2(x)}, \underbrace{\Phi(Y, H, s * x)}_{\Phi_3(x)}.$$

Then, by lemma 5.4 (vi), $\Omega(\lambda x. \Phi_1(x)) \text{ maj } \Omega(\lambda x. \Phi_2(x))$, $\lambda x. \Phi_3(x)$, and by lemma 5.4 (i),

$$\underbrace{H^*(\Omega(s), \Omega(\lambda x. \Phi_1(x)))}_{\hat{H}(s, \lambda x. \Phi_1(x))} \text{ maj } \underbrace{H(\Omega(s), \Omega(\lambda x. \Phi_2(x)))}_{\hat{H}(s, \lambda x. \Phi_2(x))}, H(s, \lambda x. \Phi_3(x)),$$

which, by lemma 5.4 (v), implies

$$\Omega(s @ \hat{H}(s, \lambda x. \Phi_1(x))) \text{ maj } \Omega(s @ \hat{H}(s, \lambda x. \Phi_2(x))), s @ H(s, \lambda x. \Phi_3(x)),$$

and finally, by lemma 5.5 (i) and (iii),

$$\underbrace{\hat{Y}^*(s @ \hat{H}(s, \lambda x. \Phi_1(x)))}_{\Phi^*(Y^*, H^*, s)} \geq \underbrace{\hat{Y}(s @ \hat{H}(s, \lambda x. \Phi_2(x)))}_{\Phi^*(Y, H, s)}, \underbrace{Y(s @ H(s, \lambda x. \Phi_3(x)))}_{\Phi(Y, H, s)}.$$

\dashv

COROLLARY 5.8 (to the proof). *Given Y, H (of appropriate types), there exists a functional $\Phi^* : \rho^* \rightarrow o$ such that for any solution Φ of equation (3) we have $\forall s (\Phi(s) \leq \Phi^*(s))$.*

PROOF. Let Y and H be fixed. Moreover, assume $\Phi(Y, H)$ satisfies equation (3), for all s . By lemma 5.5 (ii), the functional \hat{Y} is continuous (\hat{Y} as defined in the proof theorem 5.7). Therefore, there exists a unique functional Φ^* satisfying equation $\Phi^*(s) = \hat{Y}(s @ \hat{H}(s, \lambda x. \Phi^*(s * x)))$. We have just shown that for all s , $\Phi^*(s) \geq \Phi(s)$. \dashv

Notice that the result of corollary 5.8 does not apply to equation (2), e.g. take

$$Y(\alpha) = \begin{cases} 1 & \text{if } \exists n \forall m (\alpha(m) = 1) \\ 0 & \text{otherwise,} \end{cases}$$

and $H(s, \beta) = \beta(0)$, it is not true that there is a bound on all possible solutions. This together with the theorem above strongly indicates that equation (2) cannot be defined from modified bar recursion.

Lemma 5.2 states that for any given α , at some point n , the value of $Y(\bar{\alpha}n @ \beta)$ is bounded by n , for arbitrary β . Let α be fixed. Moreover, let

$$k = \mu n (\forall \beta \in \bar{\alpha}n (Y(\alpha) \leq n)),$$

(which by lemma 5.2 is well defined) and $K := \{0, \dots, k\}$. If the functional Y received two arguments s and $H(s, \lambda x. B(s * x))$ (instead of their overwrite) equation (3) would read as, $B(s) = Y(s, H(s, \lambda x. B(s * x)))$, or simply,

$$(*) B(s) = Y(s, \lambda x. B(s * x)).$$

Now, still considering the path α , for any s which extends $\bar{\alpha}n$ we would have that $Y_s : (\rho \rightarrow K) \rightarrow K$. If we can show that (*) has a solution (for those particular s extending $\bar{\alpha}n$) then we could derive that equation 3 always has a solution in \mathcal{M} . This follows by bar induction, simply because the bar condition would be proven by the argument above (i.e. in any path α there exists an n such that the equation reduces to (*), and therefore has a solution) and once we eventually have a solution in each branch we can give a solution for the equation on all arguments. In the following we show some special cases where equation (*) always has a solution.

CLAIM 5.9. *If Y does not depend on s then equation (*) always has a solution.*

PROOF. If Y does not depend on s we consider the value of Y on $\lambda x. k$, for each $k \in K$. It is clear that a cycle exists, i.e. there are k_0, \dots, k_n s.t. $k_i = Y(\lambda x. k_{(i+1) \bmod n})$. We then define $B(s) := k_{(|s| - n) \bmod n}$. \dashv

If Y depends on s a solution is guaranteed if ρ is a singleton.

CLAIM 5.10. *Let K be a finite set and $Y : (o, K) \rightarrow K$ an arbitrary function. Equation $B(n) = Y(n, B(n + 1))$ always has a solution.*

PROOF. For each point n and value $k \in K$, let $\langle l_0, \dots, l_n \rangle$ (where $l_i \in K$ and $l_n = k$) be such that $Y(i - 1, l_i) = l_{i-1}$, $0 < i \leq n$. Consider the tree of all such paths, such tree is finitely branching. We can also find a branch of arbitrary length, just pick a $n \in \mathbb{N}$ and a $k \in K$ and work our way to the root. Therefore, by König lemma there is an infinite path. \dashv

The result above can be extended to the case when ρ is a finite set (which still gives a finitely branching tree) using the same argument. For instance $B(s) = Y(s, B(s * 0), B(s * 1))$ has always a solution. The argument breaks down for an infinitely branching tree since König's lemma cannot be applied. The reason is that, in a infinitely branching tree it can be the case that all branches are finite but the whole tree is still infinite. This would strongly indicate that equation (*) indeed does not always have a solution for some very clever Y which, as s grows, makes the branches terminate. Such argument for the non-existence of solutions for (*), however, would not apply to the original problem since the Y there does not have any information on the s .

The problem whether equation (3) (for fixed $Y, H \in \mathcal{M}$) always has a solution, for $\rho \geq o$, remains open. In the following we comment on other open problems regarding modified bar recursion .

§6. Open problems. In order to give a program for the fan functional we used two schemas of bar recursion (2) and (3). One could ask,

- Can the fan functional be defined by equation (2) or (3) alone?

The question could be answered affirmatively if one schema can be primitive recursively defined from the other. This raises the following questions,

- What is the relation between modified bar recursion (3) and bar recursion (2)?

Since equation (2) is strictly stronger than Spector's original definition (1), and this separation is obtained via the model \mathcal{M} , we ask,

- Is there a functional Φ (in the full type structure) which, for parameters Y, H, s in \mathcal{M} , satisfies equation (3)?

We have shown that if this question is answered affirmatively then Φ has to belong to \mathcal{M} , which implies that (3) cannot be used to define (2). On the other hand, if the question is answered negatively then we can conclude that (1) cannot be used to define (3).

§7. Conclusion. In this paper we discussed modified bar recursion a variant of Spector's bar recursion that seems to be of some significance in proof theory and the theory and higher type recursion theory. Our main result was an abstract modified realizability interpretation (where realizability for falsity is uninterpreted) of the axioms of countable and dependent choice that can be used to extract programs from non-constructive proof using this axiom. A similar result is in [2], but we claim that our solution more accessible and, in a sense, more 'civilized'. It can be noted here that the weak form of modified bar recursion (4) used for the realization of dependent choice can be implemented quite efficiently by equipping the functional with an internal memory that records the value of $H(s, \lambda x. \Phi(s * x))$ (which is of type o) and thus avoids its repeated computation. Such an optimization does not seem to be possible for the (allegedly more efficient) solution given in [2]. In order to make the realizability interpretation of dependent choice useful for program synthesis seems necessary to combine it with optimizations of the A-translation as development e.g. in [4] and [5]. To find out whether this is possible will be a subject of further research.

Another important result was a definition of the fan functional using modified bar recursion and a version of bar recursion due to Kohlenbach, improving [3] and [16] where a PCF definition of the fan functional was given. In [20] this definition of the fan functional has been applied to give a purely functional algorithm for exact integration of real functions.

The paper concluded with some new results on the model \mathcal{M} of strongly majorizable functionals which may give us some direction in future research on the open problem whether modified bar recursion exists in \mathcal{M} .

Acknowledgements. We would like to thank Ulrich Kohlenbach for pointing out some mistakes in an early formulation of section 5, and also for suggesting corrections.

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