Green-Tao numbers and SAT

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SAT 2010
Edinburgh, July 12, 2010

Avoiding progressions of size 3 in the primes

Partition $\{2, 3, 5, 7, 11, 13, 17, 19, 23\}$
into two parts
such that no part
contains an arithmetic progression of size 3:

2 3 5 7 | 11 13 17 19 23

2 3 5 11 | 7 13 17 19 23

2 3 5 11 13 | 7 17 19 23

$$\mathbf{grt_2(3,3)} > 9$$

$$\mathrm{grt}_2(3,3) = 23$$

The **Green-Tao Theorem** guarantees that
$\mathrm{grt}_m(k_1, \ldots, k_m) \in \mathbb{N}$ always exists.

The basic SAT translation

`OKlibrary` at `http://www.ok-sat-library.org`

```
(%i1) oklib_load_all();
(%i2) output_greentao2_stdname(3,9);
> more GreenTao_2-3_9.cnf
c Green-Tao problem (diagonal form), created by the OKlibrary:
c 2 parts, arithmetic progressions of size 3, and 9 prime numbers.
c Variables and associated prime numbers:
c 1 : 2
c 2 : 3
c 3 : 5
c 4 : 7
c 5 : 11
c 6 : 13
c 7 : 17
c 8 : 19
c 9 : 23
p cnf 9 14
2 3 4 0
2 4 5 0
3 5 7 0
2 5 8 0
4 6 8 0
2 6 9 0
5 7 9 0
-4 -3 -2 0
-5 -4 -2 0
-7 -5 -3 0
-8 -5 -2 0
-8 -6 -4 0
-9 -6 -2 0
-9 -7 -5 0
```

Overview

In this initial phase (see [5, 6]) of the investigations into

Ramsey theory and SAT

we

- computed “the” basic Green-Tao numbers, and
- determined the “best” available SAT methods.

We investigated

1. how to get more problems of non-astronomical size
2. how to translate the non-boolean problems (i.e., $m > 2$) into boolean problems.

General methods were found to approach both problems.

In the future we hope to establish Ramsey-type problems
as an important class of benchmark problems
for (general) SAT solvers.

Outline

On the history of the Green-Tao theorem

- Arithmetic progressions in the primes have been investigated for more than 200 years.
- The existence of arbitrarily long arithmetic progressions in the primes is a special case of the famous “k-tuple conjecture” of Hardy and Littlewood in 1923 (still wide open).
- The final proof in 2004 by Ben Green and Terence Tao is based on *Additive Number Theory* and *Ramsey Theory*.

Simple parameter tuples

Already $\mathrm{grt}_1(k)$ for $k \in \mathbb{N}$ poses a non-trivial mathematical problem, namely

> the question is to find the smallest $n \in \mathbb{N}$
> such that the first n prime numbers
> contain an arithmetic progression of length k.

- Trivially $\mathrm{grt}_1(1) = 1$ and $\mathrm{grt}_1(2) = 2$, while $\mathrm{grt}_1(3) = 4$ (confirmed by the progression $(3, 5, 7)$).
- The computation of $\mathrm{grt}_1(k)$ has nothing to do with SAT solving, so we can't contribute here, but the known values give a first feeling for the growth involved.

The known values of $\mathrm{grt}_1(k)$

k	$\mathrm{grt}_1(k)$
1	1
2	2
3	4
4	9
5	10
6	37
7	155
8	263
9	289
10	316
11	21,966
12	23,060
13	58,464
14	2,253,121
15	9,686,320
16	11,015,837
17	227,225,515
18	755,752,809
19	3,466,256,932
20	22,009,064,470
21	220,525,414,079

We see that for the exploration of non-simple parameter tuples only $k \leq 10$ is feasible.

Core parameter tuples

a b	3	4	5	6	7
3	23	79	528	≥ 2072	> 13800
4	-	512	> 4231		
5	-	-	≥ 34309		

a, b c	3	4	5
$3, 3$	137	≥ 434	> 1989
$3, 4$	-	> 1662	> 8200

a, b, c d	3	4
$3, 3, 3$	> 384	> 1052
$3, 3, 4$	-	> 2750

Regarding the solvers

1. All lower bounds are computed by local-search algorithms.
2. While in the 5 successful cases of computing (core) Green-Tao numbers, for determining unsatisfiability a DPLL-like (complete) SAT solver is used.
3. Already in these cases we see a good variety of algorithms/solvers being best:
    - `OKsolver-2002` (look-ahead) versus `minisat2` (conflict-driven) on the complete side,
    - `adaptnovelty+` for the binary lower bounds,
    - except of $(5, 5)$ where `survey propagation` succeeded,
    - and `rnovelty+` for the non-binary lower bounds.

We need to discuss what to do in the non-binary cases!

Regarding growth

1. Regarding $\mathrm{grt}_1(k)$: Proven is an exponential tower of height 8, conjectured
$$\mathrm{grt}_1(k) \le \pi(k! + 1).$$
2. We have furthermore
$$\mathrm{grt}_m(k_1, \dots, k_m) \le \mathrm{grt}_1(\mathrm{vdw}_m(k_1, \dots, k_m)).$$
3. We conjecture that this yields the right order of magnitude, that is,

   GT-numbers are (just) exponential in
   van-der-Waerden numbers.

We need more food (for the solvers)!

Cases with $k_i = 2$

We call a parameter-tuple **core**, if

- it has at least two entries
- and every entry is at least 3.

Parameter-values 2 seems not to have been systematically considered until now:

Tuples of the form $(2, \dots, 2, k_1, \dots, k_l)$ we call **transversal extensions** of $(k_1, \dots, k_l)$.

Finding a solution for m initial 2's means, that we can discard at our will m prime numbers.

Linear growth

Keeping $(k_1, \dots, k_l)$ and growing the prefix of 2's, we prove in the underlying [5, 6] (in a general framework):

Theorem *If we have the Szemerédi property, then for every tuple* $(k_1, \dots, k_l)$*, for every* $\varepsilon > 0$ *and for m big enough we have:*

$$N_{m+l}(2, \dots, 2, k_1, \dots, k_l) \leq (1 + \varepsilon) \cdot m.$$

Since the Green-Tao theorem actually proves the Szemerédi property, we get

$$\mathrm{grt}_{m+l}(2, \dots, 2, k_1, \dots, k_l) \leq (1 + \varepsilon) \cdot m.$$

Remarks on transversal GT-numbers

- Now the conflict-driven solvers become superior.
- Regarding the lower bounds, the `novelty`-family is no longer dominant, but other local-search algorithms often perform best (`saps`, `sapsnr`, `rsaps`, `walksat`, `walksat-tabu`).
- See the paper/report for the numbers/bounds.

Non-boolean clause-sets

The “true” generalisation of boolean CNF to non-boolean CNF seems to be the following:

1. variables v have (finite) domains D_v
2. literals are of the form “$v \neq \varepsilon$” for some $\varepsilon \in D_v$;
3. these clauses are called “no-goods” in constraint solving.

For a systematic investigation see [2, 3, 4].

With these non-boolean clause-sets also the non-boolean Green-Tao problems, for tuples $(k_1, \dots, k_m)$, now have a canonical representation, using m values.

A general scheme for a boolean translation

The *generic boolean translation* $F \rightsquigarrow T(F)$ for a non-boolean clause-set F is as follows (using $m := |D_v|$);

- For each variable v, choose unsatisfiable variable-disjoint boolean clause-sets F_v with at least m clauses.
- Choose different clauses $C_1, \dots, C_m \in F_v$.
- Literals "$v \neq \varepsilon_i$" are replaced by the clauses C_i.
- The "remainder clauses" in $R_v := F_v \setminus \{C_1, \dots, C_m\}$ are all added to the translation.

Note that

$$n(T(F)) = \sum_{v \in \mathrm{var}(F)} n(F_v)$$

$$c(T(F)) = c(F) + \sum_{v \in \mathrm{var}(F)} c(R_v).$$

Example: The direct translations

Here we choose

$$F_v = \{ \{v_1\}, \dots, \{v_m\}, \{\overline{v_1}, \dots, \overline{v_m}\} \},$$

and we choose the unit-clauses to correspond to the values.

1. For the *weak form* (using only ALO-clauses) that's it (so we have one remainder clause).
2. For the *strong form* we add all positive binary clauses to (the remainder of) F_v (so obtaining the AMO-clauses).

The weak nested translation

Here we use $p := m - 1$ (boolean) variables $v_1, \dots, v_p$ and

$$F_v = \{ \{v_1\}, \{\overline{v_1}, v_2\}, \dots, \{\overline{v_1}, \dots, \overline{v_{p-1}}, v_p\}, \{\overline{v_1}, \dots, \overline{v_p}\} \}.$$

There are no remainder clauses.
Yet we tested these (and other, related) translations only on the Green-Tao instances, but this we did rather extensively.

Big surprise:

For “large” m the logarithmic translation was best,
and for all other m the weak nested translation —
for all solver types.

“Best” often means by orders of magnitudes.

Summary

I This concludes for us the initial phase.

II Several other forms of problems from Ramsey theory have also been considered.

III We see a rich diverse behaviour, where every solver can be best on some class for some parameters.

IV Now the task is to understand what's going on!

V Especially the unsatisfiable cases need to be improved.

VI So amongst others we will investigate tree-resolution and full-resolution complexity in detail, experimentally as well as theoretically.

VII As an aside, the resolution proofs found for the Green-Tao instances seem to have surprising (number-theoretical) regularities.

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