Computability on topological spaces via domain representations

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Abstract

Domains are ordered structures designed to model computation with approximations. We give an introduction to the theory of computability for topological spaces based upon representing topological spaces and algebras using domains. Among the topics covered are: different approaches to computability on topological spaces; orderings, approximations and domains; making domain representations; effective domains; classifying representations; type two effectivity and domains; special representations for inverse limits, regular spaces and metric spaces. Lastly, we sketch a variety of applications of the theory in algebra, calculus, graphics and hardware.

1 Introduction

The theory of topological spaces and continuous functions is about the approximation of data and the functions that preserve those approximations. Approximation is expressed by open subsets of the set of data. The primary intuitions are geometric and its original applications were in geometry, differential equations and functional analysis, where data are made from real and complex numbers, functions and operators. A century of research has made topology essential to mathematics and physical science (see Aull and Lowen [2, 3], James [41]). The question arises:

> *How does one compute with data such as real and complex numbers, functions and operators? More generally, how does one compute with data from a topological space?*

Understanding computation on topological spaces is important. For example, it is needed to improve practical computation with continuous data;

to compare and unify digital and analogue computation; to explore computation in analysis and geometry; and to establish the computational and logical nature of physical systems - to name just four research problems of contemporary interest.

There are several answers to the computability question above, some methods are specific to the real numbers, some are general for a class of topological spaces. Here we will explain one answer:

> *Represent topological spaces of data by domains and reduce computation on those spaces to computation on domains.*

We will summarise other approaches to computability shortly (Section 2).

The theory of domains and order-preserving functions is also about the approximation of data and the functions that preserve approximations. Approximation is expressed by an ordering on the set of data. The primary intuitions are computational and its primary applications are in computability theory and the semantics of programming languages and logics, where computations are defined using recursion equations on functions, memory states and environments, data, processes, formulae and types. The foundations of the subject were laid by D. S. Scott [62, 63, 64] and Yu. L. Ershov [33, 34].

Domains are ordered algebraic structures containing both approximations and the data they approximate. The ordering $\sqsubseteq$ on a domain D formulates the idea that for $a, b \in D$,

$$a \sqsubseteq b \iff \text{`datum } b \text{ is a better approximation than datum } a\text{'}.$$

The limits of sequences of such approximations are the data to be approximated. Computations are modelled as a process of finding better and better approximations.

Domains are designed to solve equations. Their orderings are used to capture some of the features of using iterative algorithms to approximate solutions. The equations are formulated as fixed point equations, i.e., for a given function $f\colon D \to D$, find a in D such that $f(a) = a$. The fixed point methods build the solutions from their approximations. The inspiration of these essential features of domains and equation solving are the *complete partially ordered set* (cpo) and the equation solving methods of the *Tarski-Knaster Fixed Point Theorem*, proved in 1927; (see, e.g., Tarski [79]). These methods found their way into computability theory via theorems such as Kleene's recursion theorems. The methods were applied on particular cpos of functions on natural numbers to explain recursion. Through the theory of domains and domain representations the wide applicability of fixed point methods to computational problems became evident.

The theory of domain representations of topological spaces is a general theory about how to:

(i) represent topological spaces using domains;

(ii) analyse computation on spaces via their representations;

(iii) compare and classify different domain representations;

(iv) compute the solutions of equations on spaces; and

(v) make applications.

In this chapter we will introduce these topics, sketch their development, and point out connections with other theories that answer the question posed above. Many kinds of domains have been discovered; we will focus on so called algebraic domains which we consider to be the most simple and useful for computability.

The structure of the chapter is this. In Section 2 we will summarise the approaches to computability and sketch their origins. In Section 3 we introduce the idea of using orderings to formulate basic ideas about approximations. This leads directly to the concept of an algebraic domain. In Section 4 we introduce the continuous functions on domains. In Section 5 we define domain representations for spaces which are the structures within which computations take place. In Section 6 we add algorithms and define what is actually computable on the approximations that make up the domain. In Section 7 we introduce some simple types of domain representations (retract, dense, etc.), and we use reductions between domains that allow us to compare representations of topological spaces and discuss the stablility or invariance of computational properties of the representations. In Section 8 we examine a special form of algebraic domain representation which we derive from K. Weihrauch's approach to computability on spaces called Type 2 Theory of Effectivity (TTE): see Weihrauch [86]. In Section 9 we look at some standard constructions of domain representations, including metric spaces. In Section 10 we sketch some applications of the theory to studies of computation on different spaces, including real numbers, local rings, Banach spaces, process algebras, distributions, etc.

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2 Computability on topological spaces: some principles, approaches and history

To compute in a topological space we will choose some representation of the space, made from a domain, and compute on the domain representation. There are other ways to compute on spaces, not all of which are equivalent, and so before examining domain representations we will view the wider technical landscape.

2.1 Principles: Concrete versus abstract computability

By a *computability theory* we mean a theory of functions and sets that are definable using a model of computation. By a *model of computation* we mean some general method of calculating the value of a function or of deciding, or enumerating, the elements of a set. The functions and sets can be made from any kind of data.

With this terminology, Classical Computability Theory on the set $\mathbb{N}$ of natural numbers is made up of many computability theories, derived from different ideas about algorithms. The fact that different computability theories lead to equivalent theories of functions and sets on $\mathbb{N}$ gives the classical theory on $\mathbb{N}$ its unity, which was epitomised by the Church-Turing Thesis and was an early discovery.

Starting in the 1940s, computability theories have been created for other sets of data, including higher types over the natural numbers, real numbers and spaces of real-valued functions. More generally, computability theories have been created for classes of structures, such as groups, rings, fields, and topological and metric spaces. However, the classification and the proofs of equivalences of models of computation - and, hence, the search for generalised Church Turing Theses and the theoretical unity they represent - have proved much more difficult to achieve for these data types. An early example of different models of computation that are of equal conceptual value but are known not to be equivalent is provided by Tait's Theorem in higher types: the fan functional on total functions is recursively continuous but not computable in Kleene's schemes S1-S9; see Normann [53].

Some general insight into the phenomenon of inequivalent computability theories is to be found by examining treatment of data in models of computation. Computability theories can be classified into two types by introducing the following concepts.

Definition 2.1. In an *abstract computability theory* the computations are independent of all the representations of the data. Computations are uniform over all representations and are isomorphism invariants.

In a *concrete computability theory* the computations are dependent on some representation of the data. Computations are not uniform, and different representations can yield different results. Computations are not isomorphism invariants.

Models of computation that are based on abstract ideas of program, equation, scheme, or logical formula are typical of abstract models. Models of computation that are based on concrete ideas of coding, numbering, or representing data using any other kind of data, are typical of concrete models. Now, the distinction of abstract versus concrete is helpful in comparing models of computation. There is a need for both abstract and concrete models and an understanding of their relationship.

Clearly, within the concrete, there is great scope for variations in models of computation and we may expect different representations to lead to different computability theories. Abstract models, too, can vary, since the choice of operations, program constructs and kinds of formulae can vary. Can there be concrete models that are sufficiently canonical to be equivalent to an abstract model?

A full general discussion of the distinction is given in Tucker and Zucker [83], motivated by their theory of abstract computation on topological spaces (see Tucker and Zucker [80, 81, 82]). The distinction is also directly relevant to the seemingly stable and unified classical world of computability on countable algebras, as pointed out in Tucker and Zucker [83]).

The theory of computable sets and functions is based on data that may be represented by finite discrete symbols. For Turing's analysis of human computation the symbols came from the set $\mathbb{B} = \{0, 1\}$ or for Kleene's theory of recursive functions the symbols were from $\mathbb{N}$. For Gödel's computations on syntax the symbols came from the set $\mathbb{N}$ of natural numbers via Gödel numberings. The early development of computability theory did not interest itself in ideas about data and how it was represented. What, after all, was worth saying about $\mathbb{B}$ and $\mathbb{N}$ other than they are *so* fundamental and an obvious place to start. In the 1950s computability theory was extended by advanced applications in logic, algebra and analysis. Studying computability now required an interest in nature of data and how it was represented because what was computable depended upon the data. In algebra, rings and fields had to be considered as structures unique up to isomorphism, not just as specific representations. In Fröhlich and Shepherdson [38] we see great attention to representations and their equivalence. In the Mal'cev-Ershov theory of numberings of countable sets and structures [49, 35, 36], representations are studied in depth: a numbering $\alpha \colon \mathbb{N} \to A$ makes explicit the idea that one chooses a numerical representation of the data in A and computes on $\mathbb{N}$. The theory of numberings plays a role in the development of domain representations.

2.2 Computability theories for topological spaces

Most computability theories for topological spaces are developed using concrete models of computation. The study of computability on the reals began with Turing in 1936, but only later was taken up in a systematic way, e.g., in Rice [59], Lacombe [47] and Grzegorczyk [39]. For example, to compute on the set $\mathbb{R}$ of real numbers with a concrete model of computation we choose an appropriate concrete representation of the set $\mathbb{R}$, such as computable Cauchy sequences.

In the case of concrete computability, there have been a number of general approaches to the analysis and classification of metric and topological structures since the 1950s:

(i) Effective metric spaces (Ceitin [18], Moschovakis [52]);

(ii) Computable sequence structures for Banach spaces (Pour El and Richards [58]);

(iii) Type 2 Theory of Effectivity or TTE (Weihrauch [85, 86]);

(iv) Algebraic domain representations (Stoltenberg-Hansen and Tucker [71, 72, 75]);

(v) Continuous domain representations (Edalat [24, 25, 26]);

(vi) Numbered spaces (Spreen [66, 67, 68, 69]).

Computable Analysis has been greatly extended over the past decade using these models, which have been seen as competing. This has made the exciting rapid growth of the subject seem messy. In fact, for certain basic topological algebras, most of these concrete computability theories have been shown to be essentially equivalent in Stoltenberg-Hansen and Tucker [77].

We ought to say a word about the abstract approach. Analysis makes heavy use of algebraic structures, such as topological groups and vector spaces, Banach spaces, Hilbert spaces, C* algebras and many more. These many sorted topological algebras specify (i) some basic continuous operations; (ii) normal forms for the algebraic representation of elements (e.g., using bases); (iii) structure-preserving operators (i.e., homomorphisms such as linear operators); (iv) approximations, through inner products, norms, metrics and topologies.

Abstract computability theories are created by simply applying the abstract models to these algebras. These can be defined by programming languages whose programs are based on the operations of the algebras. However, thanks to approximation (iv), we obtain two classes of functions: the *computable functions* and the *computably approximable functions*.

A full account of the theory on general metric algebras, together with a detailed discussion of the bridge between abstract and concrete models, can be found in Tucker and Zucker [82, 83]. The most publicised abstract computability theory for $\mathbb{R}$ is that developed in Blum, Cucker, Shub, and Smale [17], but it is a theory that does not fit the concrete models because of its use of non-effective operations such as $=$.

2.3 Domain representation theory

The idea of representing topological spaces and algebras using effective domains was, as far as we know, first made explicit in a widely circulated report Stoltenberg-Hansen and Tucker [71], later published as [72]. There a general methodology was described for topological algebras and applied to study the effective content of the completion of a computable Noetherian local ring. It was further extended to ultrametric spaces and locally compact regular spaces in [73, 74, 75] and to metric spaces in the thesis [8]. We will meet these constructions in Section 9.

A precursor to some of the central ideas of domain representability is Weihrauch and Schreiber [87], where embeddings of metric spaces into complete partial orders equipped with weight and distance are considered.

It was clear from the beginning of the development of domain theory that, in addition to the ease of building type structures, it is a theory of approximation and computation, and that computability often implies continuity. This was exploited in [34] where Ershov gave a domain representation of the Kleene-Kreisel continuous functionals. An effective and adequate domain model of Martin-Löf partial type theory is given in Palmgren and Stoltenberg-Hansen [57] which has been extended in Waagbø [84] to provide a domain representation of Martin-Löf total type theory (see also Berger [6] and Normann [54]).

3 Approximations, orderings and domains

Suppose we want to compute on a possibly uncountable structure such as the field of real numbers $\mathbb{R}$. The elements of $\mathbb{R}$ are in general truly infinite objects (Cauchy sequences or Dedekind cuts) with no finite description. However, computations that can be performed by a digital computer or Turing machine must operate on 'finite' objects. By a finite object we mean that it is finitely describable or, equivalently, coded by a natural number. In particular, the structure on which computations are to be performed must be countable. Therefore it is not possible to compute directly on $\mathbb{R}$, we can at best compute on finite approximations of elements in $\mathbb{R}$. If the approximations are such that each real number is the limit of its approximations then we can extend a computation on approximations to $\mathbb{R}$ by interpreting a computation on a real number as the 'limit' of the computations on its approximations, where such a limit exists. It follows, intuitively, that computations are continuous processes.

In this section we show that a simple analysis of the notion of approximation leads naturally to the class of *algebraic cpos*.

3.1 Approximations and orderings

Let us consider the problem of approximation abstractly. Suppose that X is a set, or more generally, a structure. To say that a set P is an approximation for X should mean that elements of P are approximations for elements of X. That is, there is a relation $\prec$, the *approximation relation*, from P to X with the intended meaning for $p \in P$ and $x \in X$,

$$p \prec x \iff \text{“}p \text{ approximates } x\text{”}.$$

We illustrate this with a few relevant examples.

Example 3.1. Let $P = \{[a,b]\colon a \le b, a, b \in \mathbb{Q}\}$ and $X = \mathbb{R}$. Define

$$[a,b] \prec x \iff x \in [a,b].$$

Note that P is countable and consists of finite elements in the sense that an interval $[a,b]$ is finitely describable from finite descriptions of the rational numbers a and b and the symbols "[", "]" and ",". Furthermore, each $x \in \mathbb{R}$ is the 'limit' (intersection) of its approximations.

Example 3.2. Let $P = \mathbb{Q}$ and $X = \mathbb{R}$. For $a \in \mathbb{Q}$ and $x \in \mathbb{R}$ define

$$a \prec x \iff a < x$$

where $<$ is the usual order on $\mathbb{R}$.

Note that Example 3.1 provides a better approximation of $\mathbb{R}$ than Example 3.2 in that $[a,b] \prec x$ gives more information than $a \prec x$.

Example 3.3. Let X be a topological space with a topological base $\mathcal{B}$. For $B \in \mathcal{B}$ and $x \in X$ define

$$B \prec x \iff x \in B.$$

Let P and X be sets and let $\prec$ be a relation from P to X. Then $\prec$ induces in a natural way a relation $\sqsubseteq$ on P, the *refinement* (*pre-*)*order* obtained from or induced by $\prec$: for $p, q \in P$ let

$$p \sqsubseteq q \iff (\forall x \in X)(q \prec x \implies p \prec x).$$

Thus $p \sqsubseteq q$ expresses that q *is a better approximation than* p, or q *refines* p, in the sense that q approximates fewer elements in X than does p. Note that the induced refinement order indeed is a *preorder*, i.e., it is reflexive and transitive.

We now put some reasonable requirements on P and $\prec$ in order to obtain an *approximation structure* for X. We require that

- each element $x \in X$ is uniquely determined by its approximations, and
- each element $x \in X$ is the 'limit' of its approximations.

In addition, for domain theoretic reasons guaranteeing the existence of fixed points, it is useful to require P to have a trivial approximation, i.e., an approximation which approximates all elements of X (and hence contains no information about elements of X). This leads us to

Definition 3.4. Let P and X be sets, $\prec$ a relation from P to X and $\sqsubseteq$ the refinement preorder obtained from $\prec$. Then $(P, \sqsubseteq)$ is an *approximation structure* for X with respect to $\prec$ if

(i) $\{p \in P\colon p \prec x\} = \{p \in P\colon p \prec y\} \implies x = y$ (*uniqueness*);

(ii) $p \prec x$ and $q \prec x \Longrightarrow (\exists r \prec x)(p \sqsubseteq r$ and $q \sqsubseteq r)$ (*refinement*);

(iii) $(\exists p \in P)(\forall x \in X)(p \prec x)$ (*trivial approximation*).

Examples 3.1 and 3.2 are approximation structures when we add a trivial approximation. Example 3.3 gives an approximation structure precisely when the space X is T_0. In this sense (i) in Definition 3.4 is a T_0 property.

3.2 Ideals and domains

Let $(P, \sqsubseteq)$ be an approximation structure for X with respect to $\prec$. Then each $x \in X$ is uniquely identified with the set $\{p \in P\colon p \prec x\}$. Note that if $p \sqsubseteq q \prec x$ then $p \prec x$. Together with (ii) and (iii) in Definition 3.4 we see that $\{p \in P\colon p \prec x\}$ is an *ideal* over $(P, \sqsubseteq)$. In particular, it is a (canonical) net 'converging' to x.

Let us recall the definitions. For a preorder $P = (P, \sqsubseteq)$, a set $A \subseteq P$ is said to be *directed* if A is non-empty and if $p, q \in A$ then there is an $r \in A$ such that $p, q \sqsubseteq r$, i.e., every finite subset of A has an upper bound in A. A subset $I \subseteq P$ is an *ideal* over P if I is directed and whenever $x \sqsubseteq y$ and $y \in I$ then also $x \in I$, that is, I is *downwards closed.*

We often use the notation $\downarrow p = \{q \in P\colon q \sqsubseteq p\}$ and $\uparrow p = \{q \in P\colon p \sqsubseteq q\}$. Note that $\downarrow p$ is an ideal, the *principal ideal* generated by p. We denote by $\mathrm{Idl}(P, \sqsubseteq)$, or just $\mathrm{Idl}(P)$, the set of all ideals over $(P, \sqsubseteq)$.

Given an approximation structure $(P, \sqsubseteq)$ of X with respect to $\prec$ we obtain an injection of X into $\mathrm{Idl}(P)$, i.e., X "lives" in $\mathrm{Idl}(P)$. In addition, $\mathrm{Idl}(P)$ contains the approximations P that we started with by means of the principal ideals $\downarrow p$. Thus

> $\mathrm{Idl}(P)$ *is a structure that contains both the original space* and *its approximations.*

$\mathrm{Idl}(P)$ is naturally ordered by inclusion $\subseteq$. For if ideals $I \subseteq J$ then J contains more approximations and hence more information about the elements approximated than does I. We consider $\mathrm{Idl}(P)$ as a structure ordered by inclusion.

Definition 3.5. Let $P = (P, \sqsubseteq)$ be a preorder. The *ideal completion* of P is the structure $\bar{P} = (\mathrm{Idl}(P), \subseteq)$.

It is easily verified that $\bar{P}$ is an *algebraic cpo* where the *compact elements* are precisely the principal ideals. We recall the definitions.

Let $D = (D, \sqsubseteq, \bot)$ be a partially ordered set with least element $\bot$. Then D is a *complete partial order* (abbreviated *cpo*) if whenever $A \subseteq D$ is directed then $\bigsqcup A$ (the least upper bound or supremum of A) exists in D. An element $a \in D$ is said to be *compact* or *finite* if whenever $A \subseteq D$ is a directed set and $a \sqsubseteq \bigsqcup A$ then there is $x \in A$ such that $a \sqsubseteq x$. The set of compact elements in D is denoted by D_c. A cpo D is *algebraic* if for each $x \in D$, the set

$$\text{approx}(x) = \{a \in D_c : a \sqsubseteq x\}$$

is directed and $x = \bigsqcup \text{approx(x)}$.

Algebraic cpos have the following representation theorem. For its simple proof see Stoltenberg-Hansen et al. [70].

Theorem 3.6. *Let $D = (D, \sqsubseteq, \bot)$ be an algebraic cpo and let $\overline{D_c}$ be the ideal completion of $D_c = (D_c, \sqsubseteq)$. Then $D \simeq \overline{D_c}$.*

Note that if D is an algebraic cpo then $(D_c, \sqsubseteq)$ is an approximation structure for D with respect to $\prec$, where for $a \in D_c$ and $x \in D$,

$$a \prec x \Longleftrightarrow a \sqsubseteq x.$$

Thus we have

Corollary 3.7. *Algebraic cpos are precisely the ideal completions of approximation structures.*

Algebraic cpos are completely determined by their sets of compact elements. Also continuous functions between algebraic cpos are completely determined by their action on compact elements. Therefore, as we shall see in Section 6, algebraic cpos carry a natural theory of effectivity by computing on the set of compact elements, and a large subclass of them are effectively closed under various constructions including the function space construction.

We say that an algebraic cpo D is *κ-based* if the cardinality of D_c equals κ, where κ is a cardinal. D is *countably based* if D_c is countable. When considering effective algebraic cpos we are thus restricted to countably based algebraic cpos.

3.3 Methodology of domain representability

Assume the task is to study computability on a set or structure X. We find a suitable set P of approximations and then form the ideal completion $\bar{P}$ of the induced approximation structure. Then $\bar{P}$ contains both the structure X and the set of approximations for X. Furthermore, the effectivity of $\bar{P}$ and hence of X is completely determined by the computability of the set P of approximations. Now we use the general theory of domains to study the structure X, including

- fixed point theorems to compute solutions to equations;
- ease in building higher type objects (e.g. streams and stream transformers, see [15], and higher type operations such as integrals and distributions);

- computability, inherited from the computability of P.

Our claim is that the use of domains (of various kinds) provides a general, uniform and useful way to study computability via approximations on a large class of structures.

4 Continuous functions and algebraic domains

Since computations are based on approximations, an approximation of the value of a computable function should depend only on an approximation of its argument. This property gives rise to a notion of continuity.

Let D and E be cpos. Then $f\colon D \to E$ is *continuous* if f is monotone and for each directed set $A \subseteq D$, $f(\bigsqcup A) = \bigsqcup f[A]$. Thus f is continuous if it preserves information and, regarding $\bigsqcup A$ as the limit of the 'net'A, preserves limits. In case D and E are algebraic then f is continuous if, and only if, f is monotone and for each $x \in D$,

$$(\forall b \in \mathrm{approx}(f(x)))(\exists a \in \mathrm{approx}(x))(b \sqsubseteq f(a)).$$

This says that for each concrete approximation b of $f(x)$ there is a concrete approximation a of x such that f applied to a 'computes' at least as much information as b.

The topology corresponding to this notion of continuity is the *Scott topology*. For an algebraic cpo it is generated by the topological base $\{\uparrow a : a \in D_c\}$.

For cpos D and E we define the *function space* $[D \to E]$ of D and E by

$$[D \to E] = \{f\colon D \to E \mid f \text{ is continuous}\},$$

and give $[D \to E]$ the pointwise ordering:

$$f \sqsubseteq g \iff (\forall x \in D)(f(x) \sqsubseteq g(x)).$$

It is easy to see that $[D \to E]$ is a cpo where for a directed set $\mathcal{F} \subseteq [D \to E]$ and $x \in D$,

$$(\bigsqcup \mathcal{F})(x) = \bigsqcup \{f(x)\colon f \in \mathcal{F}\}.$$

It is well-known that the class of algebraic cpos is *not* closed under the function space construction. The usual additional requirement (though not the finest) is to assume consistent completeness. An algebraic cpo D is said to be *consistently complete* if each consistent (i.e. bounded) pair a and b of compact elements has a supremum (denoted $a \sqcup b$). It follows that $\bigsqcup A$ exists for each consistent set $A \subseteq D$.

Definition 4.1. An *algebraic domain* is a consistently complete algebraic cpo.

Proposition 4.2. *The class of algebraic domains is closed under the function space construction.*

Let D and E be algebraic domains. Then the compact elements of $[D \to E]$ are suprema of finite consistent sets of step functions $\langle a; b\rangle$, where the latter are defined as follows for $a \in D_c$ and $b \in E_c$,

$$\langle a; b\rangle(x) = \begin{cases} b & \text{if } a \sqsubseteq x \\ \bot & \text{otherwise.} \end{cases}$$

The class of algebraic domains is quite robust in that it is closed under all the usual constructions with the exception of the Plotkin power domain construction. The category of algebraic domains along with continuous functions is cartesian closed. In addition the fixed point operator $\text{fix}\colon [D \to D] \to D$, defined by $\text{fix}(f) =$ least x such that $f(x) = x$, is continuous.

5 Domain representations

Here we define the concept of a domain representation. We begin by considering the canonical example of the reals and conclude with some comments on using different kinds of domains.

5.1 Representing the reals

Recall Example 3.1 of the interval approximation structure for the reals $\mathbb{R}$. The set of approximations P consists of all finite closed intervals with rational end points. We also add $\mathbb{R}$ to P and say $\mathbb{R} \prec r$ for each $r \in \mathbb{R}$. The induced refinement order is $[a, b] \sqsubseteq [c, d] \iff [a, b] \supseteq [c, d]$. Let $r \in \mathbb{R}$. Then, as discussed in Section 3, the ideal

$$I^r = \{[a, b] \in P : a \leq r \leq b\} \cup \{\mathbb{R}\}$$

represents r. Note that I^r has the property that $\bigcap I^r = \{r\}$. Now consider the ideal

$$I_r = \{[a, b] \in P : a < r < b\} \cup \{\mathbb{R}\}.$$

Also in this case $\bigcap I_r = \{r\}$ but $I^r \neq I_r$ in case r is a rational number. Both ideals give complete information about r and can be considered to represent r. We say that an ideal $I \in \bar{P}$ *represents* a real number r just in case $\bigcap I = \{r\}$. Let $\bar{P}^R$ be the set of ideals whose intersection is a singleton and define a function $\nu : \bar{P}^R \to \mathbb{R}$ by

$$\nu(I) = r \iff \bigcap I = \{r\}.$$

Proposition 5.1. *The function $\nu\colon \bar{P}^R \to \mathbb{R}$ is a continuous surjection with respect to the Scott and Euclidean topologies.*

We have the following picture.

$$P \hookrightarrow \bar{P} \hookleftarrow \bar{P}^R \xrightarrow{\nu} \mathbb{R}.$$

Thus, computability on $\mathbb{R}$ can be induced via the continuous function ν from computability considerations on $\bar{P}$, which in turn depends on computations on P. The tuple $(\bar{P}, \bar{P}^R, \nu)$ is a canonical example of a domain representation of $\mathbb{R}$. It is *upwards closed* in the sense that if $I \in \bar{P}^R$ and $I \subseteq J \in \bar{P}$ then $J \in \bar{P}^R$. Furthermore $\nu(I) = \nu(J)$ for such I and J. Note that if we instead choose P to consist of open intervals then $\bar{P}^R$ consisting of the ideals whose intersection is a singleton will not be upwards closed.

5.2 General definitions

We now generalise to an arbitrary topological space.

Definition 5.2. Let X be a topological space, let D be a domain and D^R a subset of D. Then (D, D^R, ν) is a *domain representation* of X in case $\nu: D^R \to X$ is a surjective continuous map when D^R is given the (relativised) Scott topology.

We have on purpose used the generic term 'domain' since the definition makes sense for any type of ordered structure or, for that matter, topological space. Commonly used ordered structures are algebraic cpos, algebraic domains, continuous domains and bifinite domains. We return to this point below and in Section 10.

Suppose we have a domain representation of a *set* X, where we thus only require ν to be a surjection. Then the domain representation induces a topology on X by giving X the quotient topology. That is, $U \subseteq X$ is open $\iff \nu^{-1}[U]$ is open in D^R. This may at times be a useful way to topologise function spaces and thus build type structures of topological spaces.

It is quite common when constructing domain representations that the obtained mapping ν is a quotient mapping. For example, this is the case for the representation of $\mathbb{R}$ given above. Thus $\mathbb{R}$ is a quotient of $\bar{P}^R$, that is

$$\mathbb{R} \simeq \bar{P}^R / \sim,$$

where $I \sim J$ if $\bigcap I = \bigcap J$.

The next step is to represent functions between and operations on topological spaces.

Definition 5.3. Let (D, D^R, ν) and (E, E^R, μ) be domain representations of X and Y respectively. A function $f: X \to Y$ is *represented* by (or *lifts* to) a continuous function $\bar{f}: D \to E$ if $\bar{f}[D^R] \subseteq E^R$ and $\mu(\bar{f}(x)) = f(\nu(x))$, for all $x \in D^R$.

Note that $\bar{f}$ is required to be defined on all of D. In certain situations when considering computability aspects it may be useful to allow partial functions on D. This is developed in Dahlgren [20].

Suppose $\bar{f}\colon D \to E$ is such that $\bar{f}[D^R] \subseteq E^R$ and such that $\nu(x) = \nu(y) \implies \mu(\bar{f}(x)) = \mu(\bar{f}(y))$. Then $\bar{f}$ induces a unique function $f\colon X \to Y$ defined by $f(\nu(x)) = \mu(\bar{f}(x))$.

Proposition 5.4. *Let (D, D^R, ν) and (E, E^R, μ) be domain representations of X and Y respectively, and assume ν is a quotient map. If $f\colon X \to Y$ is represented by a continuous function $\bar{f}\colon D \to E$ then f is continuous.*

The proposition is a trivial topological fact. The converse is more interesting. When does a continuous function $f\colon X \to Y$ have a continuous lifting $\bar{f}\colon D \to E$? We will return to this kind of questions in Section 7.

Domain representability is naturally extended to topological algebras. Recall that a topological Σ-algebra is a topological space with continuous operations specified by the signature Σ. The field $\mathbb{R} = (\mathbb{R}, +, \times, 0, 1)$ of real numbers is a relevant example here.

Definition 5.5. Let $A = (A, \sigma_1, \ldots, \sigma_n)$ be a topological Σ-algebra. Then A is *domain representable* by $D = (D, D^R, \nu; \bar{\sigma}_1, \ldots, \bar{\sigma}_n)$ when (D, D^R, ν) is a domain representation of the topological space A, and each $\bar{\sigma}_i\colon D^{n_i} \to D$ is a continuous operation on D representing the operation σ_i. The domain with operations $(D, \bar{\sigma}_1, \ldots, \bar{\sigma}_n)$ is called a Σ-*domain*.

Note that the mapping ν in the definition is a Σ-homomorphism.

5.3 Other domains

One can represent topological spaces using other kinds of domains and ordered structures. Examples are *Baire-Cantor domains* and *continuous domains*. Each yields a theory of computability on spaces with an extensive set of applications.

The Cantor-Weihrauch domains are simply the Baire and Cantor spaces of functions on $\mathbb{N}$ seen as domains; we will meet them in Section 8. K. Weihrauch created the theory of TTE computability independently of notions of domain theory. Indeed after having considered cpos as a general approximating structure, he chose to use Baire and Cantor spaces and their computability theories based on relativised Turing computability, to represent spaces. TTE has an extensive theory and a huge range of applications, see Weihrauch [86]. It is possible to view TTE as a theory of *Baire-Cantor-Weihrauch domain representations*: see Blanck [14].

The continuous domains have a different axiomatisation of the intuitions behind domains and form a larger class of structures containing the algebraic domains. They were first used for the representation of the real numbers and other topological spaces by A. Edalat, who has also created an extensive set of applications: see Section 10.9.

The relationship between the use of these various kinds of domains in representation theory has been discussed in Stoltenberg-Hansen and Tucker [77] and in Blanck [14].

6 Effectivity

In this section we impose and study notions of computability or effectivity on domains in order to study computability on the represented structure. The type of effectivity we consider is, in the terminology of Definition 2.1, concrete computability. Our computability theory is driven by the partial recursive functions. We use the Mal'cev-Ershov theory of numberings in order to extend computability from the natural numbers to other structures, such as domains.

We assume some very basic knowledge of recursion theory, that can be found in any basic text. Our notation is standard. In particular we let $\{W_e\}_{e\in\mathbb{N}}$ be a standard numbering of the recursively enumerable (r.e.) sets.

Let A be a set. A *numbering* of A is a surjective function $\alpha\colon \Omega_A \to A$, where $\Omega_A \subseteq \mathbb{N}$. It should be thought of as a coding of A by natural numbers. In case Ω_A is recursive, we say that a subset $S \subseteq A$ is *α-semidecidable* if $\alpha^{-1}(S)$ is r.e. and S is *α-decidable* if $\alpha^{-1}(S)$ is recursive.

Let B be a set with a numbering β. Then a function $f\colon A \to B$ is said to be *(α, β)-computable* if there is a partial recursive function $\bar{f}$ such that for each $n \in \Omega_A$, $\bar{f}(n)$ is defined and

$$f(\alpha(n)) = \beta(\bar{f}(n)).$$

We say that $\bar{f}$ *tracks* f.

6.1 Effective domains

At the heart of an algebraic cpo are the compact elements which play the role of the finite approximations. All computations will take place on the compact elements. Moreover continuous functions between algebraic cpos are completely determined by their behaviour on the compact elements. Thus it suffices to have a numbering of the compact elements of an algebraic cpo.

The following weak notion of effectivity suffices for many basic results with the important exception of the function space construction.

Definition 6.1. An algebraic cpo $D = (D, \sqsubseteq, \bot)$ is *weakly effective* if there is a numbering

$$\alpha\colon \mathbb{N} \to D_c$$

of D_c such that the relation $\alpha(n) \sqsubseteq \alpha(m)$ is a recursively enumerable relation on $\mathbb{N}$ (i.e., the relation $\sqsubseteq$ is α-semidecidable).

We denote an algebraic cpo D, weakly effective under a numbering α, by (D, α).

Computable elements of a weakly effective cpo are those that can be effectively approximated and effective functions are those whose values can be effectively approximated from effective approximations of the arguments.

Making this precise, given weakly effective (D, α) and (E, β), we say that an element $x \in D$ is *α-computable* if the set $\text{approx}(x)$ is α-semidecidable. The set of computable elements in (D, α) is denoted by $D_{k,\alpha}$.

A continuous function $f\colon D \to E$ is *(α, β)-effective* if the relation $b \sqsubseteq f(a)$ is α-semidecidable on $D_c \times E_c$. The intuition for the latter is that the approximations of $f(x)$ are generated effectively and simultaneously with the approximations of x. (Recall the characterisation of a continuous function between algebraic cpos from Section 4.)

It is straight forward to show that an effective function takes a computable element to a computable element and that the composition of effective functions is effective.

The set $D_{k,\alpha}$ has a natural numbering.

Theorem 6.2. *Let (D, α) be a weakly effective algebraic cpo. Then there is a numbering $\bar{\alpha}\colon \mathbb{N} \to D_{k,\alpha}$ such that*

(i) *the inclusion mapping $\iota\colon D_c \to D_{k,\alpha}$ is $(\alpha, \bar{\alpha})$-computable;*

(ii) *the relation $\alpha(n) \sqsubseteq \bar{\alpha}(m)$ is r.e., i.e.,* $\text{approx}(\bar{\alpha}(m))$ *is α-semidecidable uniformly in m; and*

(iii) *there is a total recursive function h such that for each e,*

$$\bar{\alpha}[W_e] \textit{ directed} \implies \bar{\alpha}h(e) = \bigsqcup \bar{\alpha}[W_e].$$

A numbering satisfying (i) and (ii) of the theorem is said to be a *constructive numbering* of $D_{k,\alpha}$. It is *recursively complete* if it also satisfies (iii). It is a fact that all recursively complete constructive numberings of $D_{k,\alpha}$ are *recursively equivalent* as numberings. In general, two numberings μ and ν of a set A are recursively equivalent if $\text{id}\colon A \to A$ is (μ, ν)-computable and (ν, μ)-computable.

To relate our domain theoretic notions to classical recursion theory let $\mathcal{P}$ be the algebraic domain of all partial functions from $\mathbb{N}$ into $\mathbb{N}$ ordered by graph inclusion. Let α be a standard numbering of the set $\mathcal{P}_c$ of finite functions. Then $\mathcal{P}_{k,\alpha}$ is the set of partial recursive functions. The numbering $\bar{\alpha}$ is a standard numbering of the partial recursive functions in the sense of Hartley Rogers in that it satisfies the universal property and the s-m-n theorem.

Given weakly effective algebraic cpos (D,α) and (E,β) we have the notion of an (α,β)-effective function from D to E and of an $(\bar{\alpha},\bar{\beta})$-computable function from $D_{k,\alpha}$ to $E_{k,\beta}$. They are related by the following deep theorem due to Ershov [34], a generalisation of the Myhill-Shepherdson theorem.

Theorem 6.3. *Let (D,α) and (E,β) be weakly effective domains and let $f\colon D_{k,\alpha} \to E_{k,\beta}$. Then f is $(\bar{\alpha},\bar{\beta})$-computable if, and only if, there is an (α,β)-effective function $\bar{f}\colon D \to E$ such that $\bar{f}\restriction D_{k,\alpha} = f$.*

For the function space construction a stronger form of effectivity is needed.

Definition 6.4. An algebraic domain $D = (D,\sqsubseteq,\bot)$ is *effective* if there is a numbering $\alpha\colon \mathbb{N} \to D_c$ such that the following relations are α-decidable for $a,b,c \in D_c$:

(i) $a \sqsubseteq b$;

(ii) $\exists d \in D_c(a, b \sqsubseteq d)$; and

(iii) $a \sqcup b = d$.

Proposition 6.5. *The category of effective domains with effective functions as morphisms is cartesian closed.*

The proof uses the intuitively effective criterion for determining whether a finite set of step functions is consistent, namely

$$\{\langle a_1; b_1\rangle, \ldots, \langle a_n; b_n\rangle\} \quad \text{is consistent in} \quad [D \to E]$$

if, and only if,

$$\forall I \subseteq \{1,\ldots,n\}(\{a_i : i \in I\} \text{ consistent } \Longrightarrow \{b_i\colon i \in I\} \text{ consistent}).$$

6.2 Effective domain representations

The method we pursue to study effective properties of a topological algebra A is to find an *effective* domain D representing A in the sense of Definition 5.2 and then measure the effectivity of A by means of the effectivity of the representing domain D. Thus, the effectivity of A is dependent on the domain representation D and its effectivity. In practice, as described in Section 3.3, given an algebra A one finds a *computable* or *effective* structure P of approximations for A which is such that the ideal completion $\bar{P}$ of the approximation structure P is a domain representation of A.

Definition 6.6. Let X be a topological space. Then X is (*weakly*) *effectively domain representable* by (D, D^R, ν, α) when (D, D^R, ν) is a domain representation of X and (D,α) is a (weakly) effective domain.

The computable elements of X are induced by the computable elements of D. More precisely, the set $X_{k,\alpha}$ of *computable elements* of X is the set

$$X_{k,\alpha} = \{x \in X : \nu^{-1}(x) \cap D_{k,\alpha} \neq \emptyset\}.$$

The above notions are easily extended to topological Σ-algebras. Let $(A, \sigma_1, \ldots, \sigma_n)$ be a topological Σ-algebra. Then $(D, D^R, \nu, \alpha; \bar{\sigma}_1, \ldots, \bar{\sigma}_n)$ is a (weakly) effective domain representation of $(A, \sigma_1, \ldots, \sigma_n)$ if the operations $\bar{\sigma}_i$ are α-effective, and $(D, D^R, \nu; \bar{\sigma}_1, \ldots, \bar{\sigma}_n)$ is a domain representation of $(A, \sigma_1, \ldots, \sigma_n)$.

A Σ-algebra A is said to have a *numbering with recursive operations* if there is a numbering $\beta : \Omega_A \to A$, such that each operation in A is β-computable. And we say that (A, β) is a *numbered algebra with recursive operations* if β is a numbering of A with recursive operations. Note that we put *no* requirement on the complexity of the code set Ω_A nor on the (relative) complexity of the equality relation.

Proposition 6.7. *Let $(A, \sigma_1, \ldots, \sigma_q)$ be a topological Σ-algebra weakly effectively domain representable by $(D, D^R, \nu, \alpha; \bar{\sigma}_1, \ldots, \bar{\sigma}_q)$.*

(i) *$A_{k,\alpha}$ is a subalgebra of A.*

(ii) *$A_{k,\alpha}$ is a numbered algebra with recursive operations with a numbering $\tilde{\alpha}$ induced by α.*

The first part of the proposition follows immediately since an effective domain function takes computable elements to computable elements. For the second part let $\Omega_A = \bar{\alpha}^{-1}(D_{k,\alpha} \cap D^R)$ and define $\tilde{\alpha}\colon \Omega_A \to A_{k,\alpha}$ by

$$\tilde{\alpha}(n) = \nu(\bar{\alpha}(n))$$

for $n \in \Omega_A$, where $\bar{\alpha}$ is the canonical numbering of $D_{k,\alpha}$ obtained from α as in Theorem 6.2.

Finally we introduce two notions of effectivity for functions between weakly effectively domain representable topological spaces.

Definition 6.8. Let A and B be topological spaces, weakly effectively domain representable by (D, D^R, ν, α) and (E, E^R, μ, β), respectively.

(i) A continuous function $f\colon A \to B$ is said to be (α, β)-*effective* if there is an (α, β)-effective continuous function $\bar{f} : D \to E$ representing f, that is $\bar{f}[D^R] \subseteq E^R$ and for each $x \in D^R$, $f(\nu(x)) = \mu(\bar{f}(x))$.

(ii) A function $f\colon A_{k,\alpha} \to B_{k,\beta}$ is $(\tilde{\alpha}, \tilde{\beta})$-*computable*, where $\tilde{\alpha}$ and $\tilde{\beta}$ are the numberings obtained in Proposition 6.7, if there is a partial recursive function $\tilde{f}$ such that $\Omega_A \subseteq \mathrm{dom}(\tilde{f})$ and for all $n \in \Omega_A$,

$$f(\tilde{\alpha}(n)) = \tilde{\beta}(\tilde{f}(n)),$$

that is $\tilde{f}$ *tracks* f with respect to $\tilde{\alpha}$ and $\tilde{\beta}$.

It is not difficult to see from Theorem 6.3 that if $f: A \to B$ is (α, β)-effective then $f{\restriction}A_{k,\alpha}: A_{k,\alpha} \to B_{k,\beta}$ is $(\tilde{\alpha}, \tilde{\beta})$-computable (and continuous). The converse direction is more difficult. It is related to the Kreisel-Lacombe-Shoenfield theorem [43] and Ceitin's theorem [18]. Note that continuity is not assumed in (ii).

7 Classes of domain representations

Recall that we only required of a domain representation (D, D^R, ν) of a space X that the function $\nu: D^R \to X$ be continuous. In most cases, but not all, we have stronger representation. Here are some common and useful additional properties.

Definition 7.1. Let (D, D^R, ν) be a domain representation of the topological space X.

(i) The representation is a *quotient representation* if ν is a quotient map.

(ii) The representation is a *retract representation with respect to* $\rho: X \to D^R$ if ρ is continuous and $\nu\rho = \mathrm{id}_X$.

(iii) The representation is a *homeomorphic representation* if ν is a homeomorphism.

It is straight forward to see that (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i). Recall that if we restrict ourselves to quotient representations then representable functions are continuous. For a retract representation (D, D^R, ν) with respect to ρ of X we have that $\rho\nu$ is a retract, that is, $(\rho\nu)^2 = \rho\nu$, and hence that $(D, \rho\nu[D^R], \nu \restriction \rho\nu[D^R])$ is a homeomorphic representation of X.

Consider the standard representation of $\mathbb{R}$ obtained from the approximations in Example 3.1. It is easy to see that this is a retract representation with respect to the function sending each $r \in \mathbb{R}$ to the ideal $I_r = \{[a, b] \in P : a < r < b\} \cup \{\mathbb{R}\}$. Thus we obtain a homeomorphic representation of $\mathbb{R}$. Note, however, that this representation is not upwards closed. In fact, there is no homeomorphic domain representation (D, D^R, ν) of $\mathbb{R}$ where D^R is the set of maximal elements of an algebraic domain D. (There is, however, a homeomorphic continuous domain representation consisting of the maximal elements, see Section 10.9).

Theorem 7.2. *Every T_0 topological space X has a homeomorphic algebraic domain representation.*

The construction is as follows. Let $\mathcal{B}$ be a topological base of non-empty sets closed under finite intersections as in Example 3.3. Taking the ideal completion of the approximation structure $(\mathcal{B}, \supseteq)$ and letting the representing ideals be $I_x = \{B \in \mathcal{B} : x \in B\}$, we obtain a homeomorphic representation. Thus every T_0 space X has a homeomorphic κ-based domain representation, where κ is the *weight* of X, that is, the smallest infinite cardinality of a topological base for X. In particular, each second countable T_0-space has a countably based homeomorphic domain representation. However, it is not the case that countably based domain representations are restricted to second countable spaces. As we shall see in Section 10.6 there are good effective and hence countably based domain representations of important spaces that are not second countable.

The set D^R in a domain representation (D, D^R, ν) is often referred to as the set of *total* elements in the sense that its elements give total information about the elements of the represented space. There is an abstract theory of *domains with totality*, i.e., pairs (D, D^t) where $D^t \subseteq D$ and (often) satisfies some trivial properties. We will not pursue this theory here but we use the concept.

We will mainly restrict ourselves to dense representations. A domain with totality (D, D^R) is *dense* if D^R is dense in D with respect to the Scott topology. And a domain representation (D, D^R, ν) of a space X is *dense* if (D, D^R) is dense.

The advantage of a dense representation (D, D^R, ν) is the relative ease with which a continuous function from D^R can be lifted or extended to the whole of D. It is always possible to obtain an equivalent dense representation from any given representation (D, D^R, ν) by considering the domain generated by all compact approximations lying below some element of D^R. This construction, however, is in general far from being effective. One way to deal with this problem is to use *partial* continuous functions [20]. There are important situations where liftings can be achieved also for non-dense representations [14, 45, 56].

Definition 7.3. Let $D = (D, D^R, \nu)$ and $E = (E, E^R, \mu)$ be domain representations of a topological space X. The representation D *reduces* (continuously) to E, denoted by $D \leq E$, if there is a continuous function $\phi\colon D \to E$ such that $\phi[D^R] \subseteq E^R$ and $(\forall x \in D^R)(\nu(x) = \mu\phi(x))$, i.e., ν *factors* through μ via ϕ on the representing elements D^R. We say that $D \equiv E$ when $D \leq E$ and $E \leq D$.

Like the definition of domain representability this notion of reducibility works with many types of ordered structures.

Let $\mathcal{C}$ be a class of domains with totality. We will in this connection say, e.g., that $\mathcal{C}$ is the class of dense algebraic cpos, thus suppressing the 'with totality'. Then we let $\mathbf{Spec}_{\mathcal{C}}(X)$ denote the equivalence classes of $\equiv$ over the class of domain representations (D, D^R, ν) of X, where $(D, D^R) \in \mathcal{C}$. Note that if $\mathcal{C}$ is the class of dense algebraic domains, then $\mathbf{Spec}_{\mathcal{C}}(X)$ contains a largest element, assuming X is a T_0-space, by considering the homeomorphic representation obtained from a topological base.

Theorem 7.4. (Blanck [14]) *Let $\mathcal{C}$ be the class of dense algebraic domains and assume X is a T_0-space. Then the largest degree of $\mathbf{Spec}_{\mathcal{C}}(X)$ contains precisely the retract representations of X over $\mathcal{C}$.*

In particular we know that the standard representation of $\mathbb{R}$ is a largest representation over dense algebraic domains and also equivalent to the standard homeomorphic representation of $\mathbb{R}$. In fact, Blanck shows that if (D, D^R, ν) is a retract algebraic domain representation of X then (D, D^R, ν) is a largest representation over the class $\mathcal{C}$ of dense algebraic cpos. This then applies to the standard representation of $\mathbb{R}$.

Another related but important concept is that of an admissible domain representation. The analogous notion for TTE was first formulated by Schröder [61], whereas Weihrauch considered a similar notion for second countable spaces.

Definition 7.5. Let $D = (D, D^R, \nu)$ be a domain representation of a topological space X. Then D is an *admissible* representation of X over a class $\mathcal{C}$ of domains with totality if whenever $(E, E^R) \in \mathcal{C}$ and $\phi\colon E^R \to X$ is continuous then there is a continuous function $\bar{\phi}\colon E \to D$ such that $\bar{\phi}[E^R] \subseteq D^R$ and for each $w \in E^R$, $\phi(w) = \nu\bar{\phi}(w)$.

Again the term 'domain' is generic. We will, as usual, restrict ourselves to algebraic cpos or algebraic domains.

We have the following relation between admissibility and the reduction ordering of domain representations.

Theorem 7.6. *Let $\mathcal{C}$ be the class of dense algebraic cpos. Then $D = (D, D^R, \nu)$ is a largest representation of X with respect to $\leq$ over $\mathcal{C}$ if, and only if, D is an admissible representation of X over $\mathcal{C}$.*

The theorem is true for any reasonable class $\mathcal{C}$. It is proved by considering a direct sum of a largest representation of X and $(E, E^R) \in \mathcal{C}$.

Admissibility has implications on the nature of the coding function of the representation. The following is observed in Hamrin [40].

Theorem 7.7. *Let D be an algebraic cpo and assume (D, D^R, ν) is an admissible domain representation of X over the class of dense algebraic cpos. Then ν is a quotient mapping.*

The key point here is that open sets can be characterised using nets of arbitrary large cardinalities. On the other hand we are primarily interested in effective representations D and hence D_c must be countable. It is therefore interesting to introduce cardinality restrictions to the notion of admissibility.

Definition 7.8. Let κ be an infinite cardinal and let $\mathcal{C}$ be a class of algebraic cpos with totality. Let $D = (D, D^R, \nu)$ be a domain representation of a topological space X. Then D is a κ-*admissible* representation of X over $\mathcal{C}$

if whenever $(E, E^R) \in \mathcal{C}$, the cardinality of E_c is less or equal to κ, and $\phi: E^R \to X$ is continuous, then there is a continuous function $\bar{\phi}: E \to D$ such that $\bar{\phi}[E^R] \subseteq D^R$ and for each $w \in E^R$, $\phi(w) = \nu\bar{\phi}(w)$.

Recall that if the coding function ν was a quotient then every representable function is a continuous. For κ-admissible κ-based representations we have a precise characterisation of the representable functions. We formulate it here for $\kappa = \omega$ so as not to introduce the notion of a κ-continuous function.

Theorem 7.9. *Let $\mathcal{C}$ be the class of dense algebraic cpos with totality or the class of dense domains with totality. Suppose that $(D, D^R) \in \mathcal{C}$ and $D = (D, D^R, \nu)$ is a countably based representation of X such that D is ω-admissible over $\mathcal{C}$. Let $E = (E, E^R, \mu)$ be a representation of Y which is ω-admissible over $\mathcal{C}$. Then a function $f: X \to Y$ is representable over D and E if, and only if, f is sequentially continuous.*

Recall that continuous functions are sequentially continuous.

Finally we mention a theorem from Hamrin [40] characterising the spaces representable by κ-admissible and κ-based domains.

We say that a topological space X has a *κ-pseudobase* if there is a family $\mathcal{B} \subseteq \wp(X)$ such that for each open set $U \subseteq X$ and each κ-net $S \to x \in U$ there is $B \in \mathcal{B}$ such that $x \in B \subseteq U$ and S is eventually in B. A κ-net is a net of cardinality at most κ. Thus a space X has an ω-pseudobase $\mathcal{B}$ if the condition holds for each open set U and each sequence $(x_n)_n$ approaching $x \in U$.

The space of test functions used in distribution theory is an example of a topological space that is not second countable but has a countable pseudobase. (See Section 10.6.)

Theorem 7.10. *A topological space X has a κ-based and κ-admissible domain representation if, and only if, X is a T_0-space and has a pseudobase of size at most κ.*

It has been shown by Schröder [61] using TTE that the category of spaces representable by ω-based and ω-admissible domains is cartesian closed. In fact, this category coincides with the category QCB consisting of topological spaces that are quotients of second countable spaces, see Menni and Simpson [50]. For $\kappa > \omega$ the question of finding a large cartesian closed category of topological spaces is unclear.

8 TTE and domain representability

An important and successful approach to computability on topological algebras and to Computable Analysis is Type 2 Theory of Effectivity, abbreviated as TTE. A large amount of work has been done using this approach by K. Weihrauch, his students and collaborators, and others. The idea is to generalise the basic definition of a numbering, replacing the natural numbers $\mathbb{N}$ with the Baire space $\mathbb{F} = \mathbb{N} \to \mathbb{N}$ and giving $\mathbb{F}$ the Baire topology, or, more generally, with Σ^ω, where Σ is a finite or countable set. Then the established computability theory on Σ^ω induces computability on the represented space via the numbering.

We will relate TTE to (effective) domain representability. For simplicity we restrict ourselves to the Baire space, leaving the simple coding necessary when going to finite Σ.

Let X be a topological space. We say that a partial surjective function $\rho\colon \mathrm{dom}(\rho) \subseteq \mathbb{F} \to X$ is a *TTE-representation* of X if ρ is continuous. An element $x \in X$ is *ρ-computable* if there is a recursive function in $\mathrm{dom}(\rho)$ such that $\rho(f) = x$.

Suppose $\eta\colon \mathrm{dom}(\eta) \subseteq \mathbb{F} \to Y$ is a TTE-representation of Y. Then a function $f\colon X \to Y$ is TTE-*representable* with respect to ρ and η if there is a continuous partial function $\bar{f}\colon \mathrm{dom}(\bar{f}) \subseteq \mathbb{F} \to \mathbb{F}$ tracking f, i.e., $f(\rho(x)) = \eta(\bar{f}(x))$ for each $x \in \mathrm{dom}(\rho)$. The function f is (ρ, η)-*effective* if there is a computable tracking function $\bar{f}$ for f.

A first observation is that the Baire space $\mathbb{F}$ naturally extends to an algebraic domain $\mathbb{B} = \mathbb{N}^{<\omega} \cup \mathbb{F}$ with the ordering $w \preceq v \iff w$ is an initial segment of v. The usual Baire topology on $\mathbb{F}$ is the subspace topology obtained from the Scott topology on $\mathbb{B}$. We call $\mathbb{B}$ the *Baire domain*.

It is well-known that each partial continuous function $f\colon \mathrm{dom}(f) \subseteq \mathbb{F} \to \mathbb{F}$ extends to a total continuous function $\bar{f}\colon \mathbb{B} \to \mathbb{B}$. Furthermore, if f is computable then $\bar{f}$ can be chosen to be effective in a uniform way from f. From these observations we have the following equivalence theorem:

Theorem 8.1. *Let $\rho\colon \mathrm{dom}(\rho) \subseteq \mathbb{F} \to X$ be a TTE-representation of X. Then $(\mathbb{B}, \mathrm{dom}(\rho), \rho)$ is an effective domain representation of X. An element $x \in X$ is ρ-computable in the TTE sense if, and only if, it is computable in the Baire domain representation sense.*

Furthermore, if $\eta\colon \mathrm{dom}(\eta) \subseteq \mathbb{F} \to Y$ is a TTE representation of Y then $f\colon X \to Y$ is TTE-representable (and effective) with respect to ρ and η in the TTE sense if, and only if, f is representable (and effective) with respect to ρ and η in the Baire domain representation sense.

For the converse reduction we have the following observation.

Lemma 8.2. *If D is a countably based algebraic domain then there is a surjective quotient map $\varphi\colon \mathbb{B} \to D$. Furthermore $\varphi[\mathbb{F}] = D$.*

Proof. Let (a_i) be an enumeration of D_c. For $w \in \mathbb{N}^{<\omega}$ we define $\varphi(w)$ as follows. Let $v \preceq w$ be the largest initial segment such that $\{a_{v(i)} : i <$ length of $w\}$ is consistent and let $\varphi(w) = \bigsqcup\{a_{v(i)} : i <$ length of $v\}$. Then φ is monotone on $\mathbb{N}^{<\omega}$ and hence extends uniquely to a continuous function on $\mathbb{B}$, which is easily seen to be a quotient. For $x \in D$ let $w \in \mathbb{F}$ be such that $\text{approx}(x) = \{a_{w(i)} : i \in \mathbb{N}\}$. Then clearly $\varphi(w) = x$. $\square$

It follows from the proof that if (D, α) is an effective domain then φ is effective, using the numbering of D_c given by α. Furthermore, $D_k = \varphi[\mathbb{F}_k]$. Thus we obtain

Theorem 8.3. *Let (D, D^R, ν, α) be an effective domain representation of X. Then there is a TTE-representation $\rho\colon \text{dom}(\rho) \subseteq \mathbb{F} \to X$ such that the sets of computable elements of X with respect to the two representations coincide.*

Proof. Let $\varphi\colon \mathbb{B} \to D$ be as in Lemma 8.2 and define $\rho\colon \varphi^{-1}[D^R] \cap \mathbb{F} \to X$ by $\rho(x) = \nu\varphi(x)$. $\square$

We now consider representable functions. Let (D, α) and (E, β) be countably based domains and let $\varphi\colon \mathbb{B} \to D$ and $\psi\colon \mathbb{B} \to E$ be the effective surjections obtained from Lemma 8.2. The following can be proved along similar lines.

Lemma 8.4. *Suppose $f\colon D \to E$ is (α, β)-effective. Then there is an effective function $\tilde{f}\colon \mathbb{B} \to \mathbb{B}$, obtained uniformly from f, such that $\tilde{f}[\mathbb{F}] \subseteq \mathbb{F}$, and for each $x \in \mathbb{F}$, $\psi\tilde{f}(x) = f\varphi(x)$.*

Theorem 8.5. *Let $D = (D, D^R, \nu, \alpha)$ and $E = (E, E^R, \mu, \beta)$ be effective domain representations of topological spaces X and Y, respectively. There are TTE-representations ρ and η of X and Y, respectively, such that if $f\colon X \to Y$ is effectively representable over (D, α) and (E, β) then f is effectively representable with respect to ρ and η.*

Proof. Let $\varphi\colon \mathbb{B} \to D$ and $\psi\colon \mathbb{B} \to E$ be the effective surjections obtained from Lemma 8.2, and let $\bar{f}\colon D \to E$ be an (α, β)-effective representation of $f\colon X \to Y$. Let $\tilde{f}\colon \mathbb{B} \to \mathbb{B}$ be the effective function obtained from $\bar{f}$ as in Lemma 8.4. Then we define $\rho\colon \varphi^{-1}[D^R] \cap \mathbb{F} \to X$ by $\rho = \nu\varphi$, and, similarly, $\eta\colon \psi^{-1}[E^R] \cap \mathbb{F} \to Y$ by $\eta = \mu\psi$. These are clearly continuous surjections and hence TTE-representations. Furthermore, for each $x \in \varphi^{-1}[D^R] \cap \mathbb{F}$,

$$f\rho(x) = f\nu\varphi(x) = \mu\bar{f}\varphi(x) = \mu\psi\tilde{f}(x) = \eta\tilde{f}(x),$$

which shows that f is effectively representable with respect to ρ and η. $\square$

A detailed analysis of the relationship between domain representability, using the category EQU of equilogical spaces [5], and TTE is given in Bauer [4]. Dahlgren [21] shows that there is an adjoint pair of effective functors taking a TTE-representation of a topological space X to an effective domain representation of X and, conversely, taking an effective domain representation of X to a TTE-representation of X.

9 Standard constructions

In this section we consider various standard ways to obtain algebraic domain representations.

9.1 Representation of inverse limits and ultrametric algebras

We introduced domain representations to analyse the computability of topological algebras. We wanted to study the completions of local rings and algebras of infinite processes. Both algebras were constructed as countable inverse limits of algebras; such limits posessed ultrametrics and were therefore topological algebras. Many algebras of interest in computing have this form. The following special construction for countable inverse limits was introduced in Stoltenberg-Hansen and Tucker [71, 72, 73].

Let $A = (A, \sigma_1, \ldots, \sigma_k)$ be a Σ-algebra and let $\{\equiv_n\}_{n\in\mathbb{N}}$ be a family of congruences on A. We say that $\{\equiv_n\}_{n\in\mathbb{N}}$ is *separating* if $n \geq m$ and $x \equiv_n y \implies x \equiv_m y$, and if $\bigcap_{n\in\mathbb{N}} \equiv_n \; = \{(x, x) : x \in A\}$.

There is an abundance of natural examples of algebras with a family of separating congruences. For a simple example, let $T(\Sigma, X)$ be the term algebra over a signature Σ and a set of variables X. Then, for $t, t' \in T(\Sigma, X)$, let $t \equiv_n t'$ if t and t' are identical up to height $n-1$, for $n \in \mathbb{N}$. Further examples will be given in Section 10.

Given a Σ-algebra A together with a family $\{\equiv_n\}_{n\in\mathbb{N}}$ of separating congruences we define a metric d on A by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 2^{-n} & \text{if } x \neq y, \text{ where } n \text{ is least s.t. } x \not\equiv_n y. \end{cases}$$

The metric d is an *ultrametric*, i.e., d satisfies the stronger triangle inequality

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}.$$

Furthermore, each operation σ on A is *non-expansive*, i.e., satisfies

$$d(\sigma(x_1, \ldots, x_n), \sigma(y_1, \ldots, y_n)) \leq \max\{d(x_i, y_i) : 1 \leq i \leq n\}.$$

Conversely, suppose (A, d) is an ultrametric algebra with non-expansive operations. Then we define a family $\{\equiv_n\}_{n\in\mathbb{N}}$ by $x \equiv_n y \iff d(x, y) \leq 2^{-n}$.

Given a Σ-algebra A with a family of separating congruences $\{\equiv_n\}_{n\in\mathbb{N}}$ we form the Σ-algebra

$$\hat{A} = \lim_{\leftarrow} A/{\equiv_n},$$

the inverse limit of the $A/{\equiv_n}$ with respect to the homomorphisms $\phi^n_m : A/{\equiv_n} \to A/{\equiv_m}$ defined by $\phi^n_m([a]_n) = [a]_m$, for $n \geq m$. Here $[a]_n$ denotes the equivalence class of a with respect to $\equiv_n$.

The inverse limit $\hat{A} = \lim_{\leftarrow} A/{\equiv_n}$ is a *completion* of A. The completion of $T(\Sigma, X)$ is the set $T^\infty(\Sigma, X)$ of all finite and infinite terms. The (metric) completion of an ultrametric algebra A with non-expansive operations is isomorphic *as topological* algebras to the inverse limit $\hat{A} = \lim_{\leftarrow} A/{\equiv_n}$, where $\equiv_n$ is obtained from the metric as above.

To construct a domain representation of $\lim_{\leftarrow} A/{\equiv_n}$ let

$$\mathcal{C} = \dot{\bigcup}\{A/{\equiv_n} : n \in \mathbb{N}\},$$

the disjoint union of the $A/{\equiv_n}$. Order $\mathcal{C}$ by

$$[a]_m \sqsubseteq [b]_n \iff m \leq n \text{ and } a \equiv_m b.$$

Let $D(A) = \bar{\mathcal{C}}$, the ideal completion of $\mathcal{C}$. Then $D(A)$ is an algebraic domain of a rather simple kind. It is a tree of height ω, where the maximal elements of the domain correspond to the infinite branches of the tree.

There is an embedding of $\hat{A} = \lim_{\leftarrow} A/{\equiv_n}$ into $D(A)_m$, the maximal elements of $D(A)$, given by $\psi(x) = \{[\phi_n(x)]_n : n \in \mathbb{N}\}$, where $\phi_n \colon \hat{A} \to A/{\equiv_n}$ is the mapping obtained from the inverse limit construction.

Let σ be a k-ary operation on A. We define $\phi_\sigma \colon D(A)_c^k \to D(A)$ by

$$\phi_\sigma([a_1]_{n_1}, \ldots, [a_k]_{n_k}]) = [\sigma(a_1, \ldots, a_k)]_{\min\{n_1,\ldots,n_k\}}.$$

Then ϕ_σ is well-defined and monotone and hence extends to a continuous function $\phi_\sigma \colon D(A) \to D(A)$ representing σ on $\hat{A}$.

As a final remark we mention that the Banach fixed point theorem for an ultrametric space A is a direct consequence of the fixed point theorem for $D(A)$.

9.2 Standard representation of regular spaces

In the previous section we described how ultrametric spaces and certain inverse limit spaces have homeomorphic domain representations using the maximal elements of the domain. However, it is an easy fact that the set of maximal elements of an algebraic domain is totally disconnected, whereas essentially all spaces used in analysis are not. If one wants to keep dealing with homeomorphic representations using maximal elements one is forced to consider a larger class of domains such as continuous cpos. Here we continue to consider the simpler structures of algebraic domains and drop the wish for a homeomorphic representation. From a computational viewpoint this is not as problematic as it may seem since the computations take place on the representing structure.

Many spaces, such as the real numbers, cannot be constructed as inverse limits. Thus we must find other constructions when representing a wider class of spaces. In Stoltenberg-Hansen and Tucker [75] we introduced the following general method to represent regular spaces.

Definition 9.1. Let X be a topological space. Then a family P of non-empty subsets of X is a *neighbourhood system* if $X \in P$ and

(i) if $F, G \in P$ and $F \cap G \neq \emptyset$ then $F \cap G \in P$; and

(ii) if $x \in U$, where U is open, then $(\exists F \in P)(x \in F^\circ \subseteq \bar{F} \subseteq U)$.

For $F \subseteq X$, F° denotes the interior of F and $\bar{F}$ denotes the closure of F. Note that (ii) forces the space X to be regular.

Examples are topological bases of non-empty open (or closed) sets of a regular space X. Another example is a sufficiently rich family of non-empty compact sets in a locally compact space. The set of approximations for $\mathbb{R}$ in Example 3.1 is a countable and effective neighbourhood system.

Let P be a neighbourhood system for X. Then $P = (P, \supseteq, X)$ is an approximation structure for X via the approximation

$$F \prec x \Longleftrightarrow x \in F.$$

Let $\bar{P}$ be the ideal completion of P. It is an algebraic domain. (Condition (i) is only used to show consistent completeness.)

An ideal $I \in \bar{P}$ *converges* to a point $x \in X$ if for every open set U containing x there is $F \in I$ such that $x \in F \subseteq U$. I converges to x is denoted by $I \to x$. Note that a converging ideal converges to a unique point for a T_1 space X (which we include in our definition of regularity).

We let $\bar{P}^R = \{I \in \bar{P} \colon I \text{ convergent}\}$ and define $\nu \colon \bar{P}^R \to X$ by

$$\nu(I) = x \iff I \to x.$$

For $x \in X$ we define the ideal I_x by

$$I_x = \{F \in P : x \in F^\circ\}.$$

Note that $I_x \to x$ and that $J \to x \Longleftrightarrow I_x \subseteq J$.

Theorem 9.2. *Let X be a regular space and P a neighbourhood system for X. Then $\bar{P}$ is an algebraic domain and $(\bar{P}, \bar{P}^R, \nu)$ is a retract representation of X.*

Proof. Suppose $U \subseteq X$ is open and $\nu(I) = x \in U$. By Definition 9.1 (ii) there is $F \in P$ such that $x \in F^\circ \subseteq \bar{F} \subseteq U$. Thus $F \in I_x$ and hence $F \in I$. Suppose $J \in \bar{P}^R$ and $F \in J$. Then, clearly $\nu(J) \in \bar{F}$, i.e., $\nu(\uparrow F \cap \bar{P}^R) \subseteq U$. (As usual F is identified with its principal ideal $\downarrow F$.) Thus ν is continuous.

Define $\eta \colon X \to \bar{P}^R$ by $\eta(x) = I_x$. Then $\nu \circ \eta = \mathrm{id}_X$. Furthermore η is continuous since for $F \in P$,

$$\eta^{-1}(\uparrow F \cap \bar{P}^R) = \{x \in X \colon F \in I_x\} = \{x \in X \colon x \in F^\circ\} = F^\circ.$$

□

Next we consider the problem of lifting continuous functions to the representing domains.

Theorem 9.3. *Let X and Y be regular spaces with neighbourhood systems P and Q, respectively. Let $(\bar{P}, \bar{P}^R, \nu)$ and $(\bar{Q}, \bar{Q}^R, \mu)$ be the domain representations of X and Y obtained from P and Q. Suppose $f\colon X \to Y$ is a continuous function. Then there is a continuous function $\bar{f}\colon \bar{P} \to \bar{Q}$ such that for all $I \in \bar{P}^R$,*

$$\mu(\bar{f}(I)) = f(\nu(I)),$$

i.e., $\bar{f}$ is a lifting or representation of f.

Proof. Given continuous $f\colon X \to Y$ define $\bar{f}\colon P \to \bar{Q}$ by

$$\bar{f}(F) = \{G \in Q\colon f[F] \subseteq G^\circ\}.$$

It is easily verified that $\bar{f}(F)$ is an ideal and that $\bar{f}$ is monotone. We also denote by $\bar{f}$ its unique continuous extension to all of $\bar{P}$. In fact for $I \in \bar{P}$,

$$\bar{f}(I) = \{G \in Q\colon (\exists F \in I)(f[F] \subseteq G^\circ)\}.$$

Suppose $I \in \bar{P}^R$ and $\nu(I) = x$. Then $I_x \subseteq I$ and hence $\bar{f}(I_x) \subseteq \bar{f}(I)$. Thus it suffices to show that $I_{f(x)} \subseteq \bar{f}(I_x)$.

Let $G \in I_{f(x)}$. Then $f(x) \in G^\circ$ and $x \in f^{-1}[G^\circ]$. But then there is $F \in I_x$ such that $F \subseteq f^{-1}[G^\circ]$. This shows that $G \in \bar{f}(I_x)$. □

In case $\bar{P}$ and $\bar{Q}$ are effective representations for X and Y, respectively, we see from the proof that the crucial point for knowing that a continuous function $f\colon X \to Y$ is effective with respect to the representations is that the relation $f[F] \subseteq G^\circ$ for $F \in P$ and $G \in Q$ is semidecidable.

The standard notion of a computable function on $\mathbb{R}$ is the one by Grzegorczyk [39]. Applying the above to the neighbourhood system P for $\mathbb{R}$ from Example 3.1 with a standard numbering we have the following theorem proved in [75].

Theorem 9.4. *A function $f\colon \mathbb{R} \to \mathbb{R}$ is computable in the sense of Grzegorczyk if, and only if, it is effective with respect to the above representation of $\mathbb{R}$.*

9.3 Representation of metric spaces

Most topological spaces of interest possess useful metrics that can define their open sets. In analysis these metrics typically come from norms whose general theory involves Banach spaces and Hilbert spaces, for example. The effective content of metric spaces was analysed early on in a constructive framework in Ceitin [18] - see also the monograph Kushner [44]. Fundamental early contributions based on computability theory are Lacombe [47] and Moschovakis [52]. Banach spaces have received special attention in Pour El and Richards [58], where the computability of linear operators was classified. The computability of homomorphisms between metric algebras in general is studied in Stoltenberg-Hansen and Tucker [78]. We will now consider effectivity in metric spaces using domain representations following Blanck [8, 9].

We say that a metric space (X, d) is *recursive* in the sense of Moschovakis if there is a numbering $\alpha: \Omega_\alpha \to X$ such that the metric $d: X \times X \to \mathbb{R}_k$ is (α, ρ)-computable, where ρ is a standard numbering of the recursive reals $\mathbb{R}$.

This is a very general definition. The difficulty from a computational point of view is that calculations with distances are limited to those possible with recursive reals. Nonetheless it is possible to give a weakly effective domain representation to (the completion of) a recursive metric space along the lines given below. We shall not pursue this here. Instead we give an alternative definition that strengthens the computability of the space while still covering important examples.

By an ordered field K we mean a field $K = (K, +, -, \times, 0, 1, \leq)$. The field K is *computable* if there is a numbering $\gamma: \mathbb{N} \to K$ such that all the operations and the relation $\leq$ (and hence $=$) are γ-computable. It is known that if K is a computable ordered field then its real closure is computable as an ordered field (Madison [48]). Furthermore, if K is archimedian then K can be computably embedded into $\mathbb{R}_k$, (Lachlan and Madison [46]).

Now we say that the metric space (A, d) is *computable* if there is a numbering $\alpha: \mathbb{N} \to A$ and a computable archimedian ordered field (K, γ) such that d takes values in K and d is $(\alpha \times \alpha, \gamma)$-computable. We extend this to a possibly uncountable metric space (X, d) by saying tha (X, d) is *effective* if there is a dense subset $A \subseteq X$ such that (A, d) is computable. Examples of effective metric spaces are the Euclidean spaces $\mathbb{R}^n$, the space $C[0, 1]$ of continuous functions $[0, 1] \to \mathbb{R}$ with the sup norm, and L^p spaces for rational $p \geq 1$.

Let (X, d) be a metric space with a dense subset A. A *formal closed ball* is a 'notation' $F_{a,r}$, where $a \in A$ and $r \in \mathbb{Q}_+$, the set of non-negative rational numbers. The formal ball is a name or syntax for a closed ball and we may write it semantically by

$$F_{a,r} = \{x \in X : d(a, x) \leq r\}.$$

Two formal balls are *consistent*, $F_{a,r} \uparrow F_{b,s}$, if $d(a, b) \leq r + s$. And $F_{a,r}$ *formally contains* $F_{b,s}$, $F_{a,r} \sqsubseteq F_{b,s}$, if $d(a, b) + s \leq r$.

A set $\{F_{a_1,r_1}, \ldots, F_{a_n,r_n}\}$ of formal balls is *permissible* if the balls are pairwise consistent and no ball is contained within another, i.e., for $1 \leq i < j \leq n$, $F_{a_i,r_i} \uparrow F_{a_j,r_j}$ and it is not the case that $F_{a_i,r_i} \sqsubseteq F_{a_j,r_j}$ or $F_{a_j,r_j} \sqsubseteq F_{a_i,r_i}$. We use the notation σ, τ for permissible sets.

Let P be the set of all permissible sets of formal balls. We need to extend the relation $\sqsubseteq$ to permissible sets:

$$\sigma \sqsubseteq \tau \Longleftrightarrow (\forall F_{a,r} \in \sigma)(\exists F_{b,s} \in \tau)(F_{a,r} \sqsubseteq F_{b,s}).$$

Note that consistency is characterised by

$$\sigma \uparrow \tau \Longleftrightarrow (\forall F_{a,r} \in \sigma)(\forall F_{b,s} \in \tau)(F_{a,r} \uparrow F_{b,s}).$$

Given consistent permissible sets σ and τ, the supremum $\sigma \sqcup \tau = g(\sigma, \tau)$ where g removes those formal balls in $\sigma \cup \tau$ formally containing others.

The following is immediate from the construction above. But note that we need to consider sets of formal balls in order to be able to compute the supremum operation.

Lemma 9.5. *If (A, d) is a computable metric space then the obtained structure $P = (P, \sqsubseteq, \uparrow, \sqcup, \perp)$ is computable with a numbering α obtained from the numbering of A.*

We now let $D = \bar{P}$, the ideal completion of P. Thus (D, α) is an effective domain.

An ideal $I \in D$ is *converging* if for any $\varepsilon > 0$ there exists $\{F_{a,r}\} \in I$ such that $r < \varepsilon$. An element $x \in \bar{A}$, the metric completion of A, is *approximated* by the ideal I if $(\forall \sigma \in I)(\forall F_{a,r} \in \sigma)(x \in F_{a,r})$. A convergent ideal I approximates exactly one element x in $\bar{A}$; we write $I \to x$. Let $D^R = \{I \in D : I \to x \in X\}$. The function $\nu \colon D^R \to X$ defined by

$$\nu(I) = x \Longleftrightarrow I \to x$$

is a quotient mapping.

In this way we have obtained an effective domain representation of $\bar{A}$ and hence of X.

Theorem 9.6. *Each effective metric space (X, d) has an effective domain representation (D, D^R, ν, α) such that the set $X_{k,\alpha}$ of computable elements in X induced by (D, D^R, ν, α) is a recursive metric space in the sense of Moschovakis.*

The situation with computable functions between effective metric spaces is more difficult. We state the following theorem which is, essentially, Theorem 3.4.33 in Blanck [8]. It uses Berger's generalisation in [6] of the Kreisel-Lacombe-Shoenfield theorem.

By a semieffective domain we mean one where the consistency relation on the compact elements need not be decidable. A semieffective domain representation of X in the theorem below is obtained by taking the dense part of a standard effective formal ball representation of X.

Theorem 9.7. *Let X and Y be effective metric spaces. Then there exists a semieffective domain representation (D, D^R, ν, α) of X consisting of permissible sets of formal balls such that together with a standard effective formal ball representation (E, E^R, μ, β) of Y, the following are equivalent.*

(i) *The function $f: X_{k,\alpha} \to Y_{k,\beta}$ is computable in the sense of Definition 6.8;*

(ii) *There is a continuous extension of f to $f: X \to Y$ that is effective with respect to the domain representations (D, D^R, ν, α) and (E, E^R, μ, β).*

Note that the function f in (i) is not assumed to be continuous. The implication (i) implies (ii) has the form of Ceitin's Theorem, that computability implies effective continuity, as a corollary.

9.4 Representation of partial and discontinuous functions

There are important phenomena in computing that are not continuous. For example, suppose we model a *stream* of data as a function from time into a set of data, where time is thought of as continuous and data is a discrete set. It is reasonable to model time by the real number line $\mathbb{R}$ or a final segment of $\mathbb{R}$ and give the data set the discrete topology. However, the only continuous functions from $\mathbb{R}$ into a discrete set are the constant functions (since $\mathbb{R}$ is a connected space). Thus transmission of discrete data in continuous time cannot be modelled by continuous functions.

Given domain representations (D, D^R, ν) of $\mathbb{R}$ and (E, E^R, μ) of the data set A, the domain $[D \to E]$ will contain approximations to arbitrary functions from $\mathbb{R}$ to A. There is no hope of having exact continuous representations of discontinuous functions. But there are best possible *approximate* representations.

Let (D, D^R, ν) and (E, E^R, μ) be domain representations of the topological spaces X and Y, respectively. Then we say that a function $f: X \to Y$ (not necessarily continuous) is *represented approximately* by (or *lifts approximately* to) $\bar{f} \in [D \to E]$ if for each $x \in D^R$,

(i) f continuous at $\nu(x) \implies \bar{f}(x) \in E^R$ and $f\nu(x) = \mu\bar{f}(x)$; and

(ii) f not continuous at $\nu(x) \implies (\exists y \in \mu^{-1}[f\nu(x)])(\bar{f}(x) \sqsubseteq y)$.

To illustrate we consider the simple example of the floor function $\lfloor\cdot\rfloor: \mathbb{R} \to \mathbb{Z}$, which is discontinuous at precisely the integer points. Let (D, D^R, ν) be the standard closed interval representation of $\mathbb{R}$ from Example 3.1. For $\mathbb{Z}$ we could have chosen the flat domain $\mathbb{Z}_\perp$. This, however, would give no information at points of discontinuities. Instead we let $E = \wp_f(\mathbb{Z}) \cup \{\mathbb{Z}\}$ ordered by reverse inclusion $\supseteq$. In fact, E is the upper (or Smyth) power domain of $\mathbb{Z}_\perp$. Letting E^R be the set of maximal elements in E, i.e., the set of singletons $\{n\}$, we obtain a domain representation (E, E^R, μ) by mapping $\{n\}$ to n.

Define $f\colon D_c \to E$ by $f([a,b]) = \{n \in \mathbb{Z} : \lfloor a \rfloor \le n \le \lfloor b \rfloor\}$ and extend f continuously to D. Then clearly f represents the floor function approximately. But note that at the discontinuity $n \in \mathbb{R}$ we have for $\nu(I) = n$ that $f(I) \subseteq \{n-1, n\}$. Thus, by choosing our representations with some care we are able to recover much information also at points of discontinuities.

If a function f has an approximate representation then it has a best approximate representation.

Theorem 9.8. ([15]) *Let (D, D^R, ν) and (E, E^R, μ) be algebraic domain representations of X and Y, respectively. Assume that D^R is dense in D, and that (E, E^R, μ) satisfies the following local property: if $x \sqsubseteq y$ and $x \in E^R$ then $y \in E^R$ and $\mu(x) = \mu(y)$. Let $f\colon X \to Y$ be a function and assume that f has one approximate representation in $[D \to E]$. Then there is a best approximate representation $\bar{f} \in [D \to E]$ in the sense of the domain ordering.*

10 Applications

A theory of computability on topological spaces can be used to analyse computation in many application areas, including analysis, algebra, semantics of data types and programming, graphics and hardware.

10.1 More on real numbers

Throughout the paper we have chosen the field $\mathbb{R}$ of real numbers together with the closed interval domain representation of Example 3.1, which we here denote by $\mathcal{R}$, as a canonical example. We have observed that this representation is an effective dense retract representation, that the elements in $\mathbb{R}$ computable from the representation are precisely the recursive reals, and that the effective functions from $\mathbb{R}$ to $\mathbb{R}$ are precisely the functions computable in the sense of Grzegorczyk. In addition, the representation is ω-admissible and it is a largest representation of $\mathbb{R}$ with respect to the reduction $\le$ from Definition 7.3.

Now consider the set $C(\mathbb{R}, \mathbb{R})$ of continuous functions from $\mathbb{R}$ to $\mathbb{R}$. This space has a natural topology, namely the compact-open topology. The set $C(\mathbb{R}, \mathbb{R})$ has a natural domain representation $[\mathcal{R} \to \mathcal{R}]$ where the representing elements $[\mathcal{R} \to \mathcal{R}]^R$ are those continuous domain functions representing functions in $C(\mathbb{R}, \mathbb{R})$. The obtained function

$$\nu\colon [\mathcal{R} \to \mathcal{R}]^R \to C(\mathbb{R}, \mathbb{R})$$

induces a topology on $C(\mathbb{R}, \mathbb{R})$ from the relativised Scott topology on $[\mathcal{R} \to$

$\mathcal{R}]^R$ which coincides with the compact-open topology (see Blanck [11]), i.e.,

$$([\mathcal{R} \to \mathcal{R}], [\mathcal{R} \to \mathcal{R}]^R, \nu)$$

is a domain representation of $C(\mathbb{R}, \mathbb{R})$. Di Gianantonio [23] used signed digit representations of real numbers to construct another domain representation capturing the compact-open topology of $C(\mathbb{R}, \mathbb{R})$.

It is well-known that $C(\mathbb{R}, \mathbb{R})$ is not locally compact and hence there is no 'natural' topology on $C(C(\mathbb{R}, \mathbb{R}), \mathbb{R})$. On the other hand the type structure over $\mathcal{R}$ is well-behaved and therefore we can construct a type structure also over $\mathbb{R}$ including $C(C(\mathbb{R}, \mathbb{R}), \mathbb{R})$ and give each such type a topology.

To make this precise we define the set of finite type symbols as follows: ι is a type symbol, and if σ and τ are type symbols then $(\sigma \to \tau)$ is a type symbol. The pure type symbols are $t_0 = \iota$ and $t_{n+1} = (t_n \to \iota)$. For each type symbol σ we define a domain with totality $(\sigma(\mathcal{R}), \sigma(\mathcal{R})^R)$. Simultaneously we define the type $\sigma(\mathbb{R})$ over $\mathbb{R}$ and a surjective map $\nu_\sigma\colon \sigma(\mathcal{R})^R \to \sigma(\mathbb{R})$ such that $(\sigma(\mathcal{R}), \sigma(\mathcal{R})^R, \nu_\sigma)$ is a domain representation of $\sigma(\mathbb{R})$.

For the base case we use the standard closed interval domain representation $(\mathcal{R}, \mathcal{R}^R, \nu)$. Thus we let $\iota(\mathcal{R}) = \mathcal{R}$, $\iota(\mathcal{R})^R = \mathcal{R}^R$, $\iota(\mathbb{R}) = \mathbb{R}$ and $\nu_\iota = \nu$.

Inductively let $(\sigma \to \tau)(\mathcal{R}) = [\sigma(\mathcal{R}) \to \tau(\mathcal{R})]$ and let $(\sigma \to \tau)(\mathcal{R})^R$ be the set of functions in $(\sigma \to \tau)(\mathcal{R})$ representing a function from $\sigma(\mathbb{R})$ into $\tau(\mathbb{R})$ via ν_σ and ν_τ. Then let $(\sigma \to \tau)(\mathbb{R})$ be the set of functions from $\sigma(\mathbb{R})$ into $\tau(\mathbb{R})$ having a representing function in $(\sigma \to \tau)(\mathcal{R})$. Finally let $\nu_{(\sigma\to\tau)}\colon (\sigma \to \tau)(\mathcal{R})^R \to (\sigma \to \tau)(\mathbb{R})$ be the map taking a representing function in $(\sigma \to \tau)(\mathcal{R})^R$ to the function in $(\sigma \to \tau)(\mathbb{R})$ that it represents.

By the fact that the category of effective algebraic domains is cartesian closed, the domain representations $(\sigma(\mathcal{R}), \sigma(\mathcal{R})^R, \nu_\sigma)$ induces a topology (the quotient topology) and effectivity on each type $\sigma(\mathbb{R})$.

D. Normann shows in [55] that each representation $(\sigma(\mathcal{R}), \sigma(\mathcal{R})^R, \nu_\sigma)$ is dense. This is analogous to the density theorem for the finite type structure over the discrete space $\mathbb{N}$ of natural numbers proved by Berger [6], but uses by necessity a different proof. Normann also observes some 'anomalies' of the type structures $\sigma(\mathbb{R})$, e.g., that the space $t_2(\mathbb{R})$ is not metrizable.

The natural continuous domain representation for real numbers is the interval domain consisting of real intervals; this suggests strong connections to Interval Analysis [51, 1]. For example, an often used notion in Interval Analysis is the *monotone interval function*, which is nothing more than a monotone map on the interval domain. Interval Analysis has traditionally used the topology induced by the Moore metric, whereas the Scott topology has been used for the interval domain. It is easy to construct interval functions that are continuous with respect to either topology but not both. In [60] it is shown that for a continuous function f the optimal interval representation of f is continuous with respect to both topologies.

Interval Analysis is an established approach to practical exact computation. The interval domain and certain substructures thereof have also been used to investigate and reason about the practical implementation of exact real arithmetic [12, 13]. Thus, domain representations can be used to reason abstractly about the computability of functions, and to model concretely the exact steps taken in making exact real computations. Thus, there is evidence that domain representations may be a powerful tool towards practical exact computation on many forms of continuous data.

10.2 Local rings

In 1983 we knew a great deal about computable algebra (see, e.g., our later survey [76]), and our interest in domain representability began with the problem of investigating the computability of local rings. Thinking about the completions of local rings, we wanted a *general* method of introducing computability into uncountable algebras. There were four algebras in view: complete local rings, algebras of infinite processes (satisfying Bergstra and Klop's laws for ACP), algebras of infinite terms, and the field of real numbers. The first three had a common structure: they were inverse limits of countably many factor algebras and looked like domains!

Let R be a local commutative Noetherian ring whose unique maximal ideal is $\mathbf{m}$. We showed that $\mathbf{m}$ is decidable when R is computable as a ring. Define for $x, y \in R$ and $n \in \mathbb{N}$,

$$x \equiv_n y \Leftrightarrow x - y \in \mathbf{m}^n,$$

which is decidable. By Krull's Theorem, $\{\equiv_n\}_{n \in \mathbb{N}}$ is a family of separating congruences with respect to the ring operations and the general constructions of Section 9.1 can be applied to obtain an effective domain representation of the completion of R.

The local ring and the general method was circulated in Stoltenberg-Hansen and Tucker [71] and later published in Stoltenberg-Hansen and Tucker [72]).

10.3 Process algebra

Think of a process made of atomic actions that can be performed sequentially or in parallel, can be independent or communicate, and can branch deterministically or non-deterministically. Such processes abound in both computers and machines, and in nature, too. In process algebra such intuitive ideas are analysed very abstractly: processes are modelled and classified by postulating operations on processes, such as

$$p \cdot q,\ p || q \text{ and } p + q,$$

and axioms that they should satisfy. There are many kinds of semantic ideas to be found in systems so there are many operations and axioms - see Bergstra and Ponse [7]. In modelling a particular system, the idea is to devise a specification that is a set of equations, based on some choice of operations. The semantics of the specification is given by solving the equations in process algebras satisfying axioms appropriate to the problem.

It is common to need complicated infinite processes in the semantic modelling of systems and so the process algebras used are complicated uncountable structures. In particular, with some process algebra methods, the algebras of infinite processes have the beautiful structure of inverse limits of finite models of equational theories. This means that algebras of infinite processes are algebras with ultrametric topologies, and the methods of Section 9.1 can be used to study processes. Applications of solving finite systems of equations in process algebras are given in [73] and infinite systems of equations in [74].

10.4 Banach spaces

Functions and functionals on $\mathbb{R}$ and $\mathbb{C}$ can be approximated in many different ways. However, the methods used have been found to have two fundamental properties in common: they use linear combinations of basic functions, and measure the accuracy of approximations by metrics derived form *norms.* Theories of these methods have been created using vector spaces equipped with norms and other operations, such as Banach spaces, Banach algebras, Hilbert spaces, and C*-algebras. Since all of these topological algebras are special kinds of metric spaces, the method for metric spaces, given in Section 9.3, can be used to make domain representations for them. Algebraic domain representations for Banach spaces were made in Stoltenberg-Hansen and Tucker [77], in order to prove the equivalence of various models of computation, including that of Pour El and Richards designed for Banach spaces.

10.5 C^∞ functions

A common way to approximate a continuous function on the real numbers is by finite collections of compact boxes enclosing the graph of the function. Tighter boxes covering a larger segment of the graph naturally yield more information about the function we wish to approximate. This idea can be generalised to approximations of C^k and C^∞ functions on the reals in a natural way: An approximation of a C^k function f is a finite set of approximations of the the function f and the first k derivatives of f (as continuous functions from $\mathbb{R}$ to $\mathbb{R}$). Similarly, an approximation of a C^∞ function f on $\mathbb{R}$ is a finite set of approximations of the function f and the first k derivatives of f *for some* $k \geq 0$.

A C^∞ function from $\mathbb{R}$ to $\mathbb{C}$ can be thought of as a pair of smooth functions from $\mathbb{R}$ to $\mathbb{R}$ (corresponding to the real and imaginary parts of the function). Thus, we can approximate smooth functions from $\mathbb{R}$ to $\mathbb{C}$ by approximating the real and imaginary parts separately. In this way we get an effective domain representation of the space of smooth functions from $\mathbb{R}$ to $\mathbb{C}$.

10.6 Test functions and distributions

An interesting class of functions in this context is the space $\mathcal{D}$ of test functions considered in distribution theory. If we restrict ourselves to one variable, a test function is simply a smooth function from $\mathbb{R}$ to $\mathbb{C}$ with compact support. Formally, the space of test functions is constructed as an inductive limit of metrisable spaces, but is itself not metrisable. In fact, it is not even first countable. Nevertheless, we may construct an effective domain representation of $\mathcal{D}$ and study computable processes on the space of test functions. This is interesting from a purely computability theoretic point of view since it has sometimes been argued that the stronger property of second countability is needed to develop a viable computability theory on a topological space (c.f. Smyth [65]). To approximate a test function f we simply add information about (i.e. bounds on) the support of f to a $C^\infty(\mathbb{R})$-approximation of f. This idea yields an ω-admissible effective domain representation of the space of test functions and thus allows us to introduce a notion of computability on $\mathcal{D}$. We note that standard operations on $\mathcal{D}$ such as integration, differentiation, regularisation, addition, and scalar multiplication are all effective with respect to this representation.

A distribution is a continuous linear functional on the space of test functions. Since we have effective representations of the spaces $\mathcal{D}$ and $\mathbb{C}$, general domain theory yields an effective domain representation of the space of distributions. Moreover, similar methods may be applied to construct effective representations of the spaces of tempered distributions and distributions with compact support. This allows us to introduce a notion of computability on the space of distributions in the spirit of Weihrauch and Zhong [88], and to study computable processes on spaces of distributions. In particular, the space of distributions, the space of tempered distributions, and the space of distributions with compact support are all effective vector spaces, the standard embedding theorems effectivise, and the Fourier transform and its inverse lift to effective functions on the space of tempered distributions. For details, see Dahlgren [22].

10.7 Volume graphics

In volume graphics, objects are defined in 3 dimensions. Objects can be regular, like buildings and crockery, or irregular but structured, like 3D body scans, or amorphous like clouds and fire. The objects may be combined to create 3D scenes. In volume graphics, objects and scenes must be created, transformed and rendered in 2D.

In practice, different objects can have quite different representations, ranging from a collection of simple mathematical functions to large 3D arrays of physical data. *Constructive volume geometry* (CVG) is a high-level approach to volume graphics that abstracts from specific representations by focussing on high level operations on volume objects. First, to unify representations, each spatial object is required to assign data, called attributes, to every point in 3D. Thus, spatial objects are modelled by vectors

$$\phi_1, \ldots, \phi_k$$

of scalar fields of the form:

$$\phi\colon \mathbb{R}^3 \to [0,1] \text{ or } \phi\colon \mathbb{R}^3 \to \mathbb{R}.$$

The attributes chosen depend on the application. For example, a simple graphics application is the RGB model which has $k = 4$ and attributes of opacity, measured by the interval $[0,1]$, and colours red (R), green (G) and blue (B), measured by $\mathbb{R}$.

Then operations on these objects are defined to make algebras of spatial objects. There are lots of simple operations to create RGB algebras, with attributes opacity and three colours. CVG algebras are as varied as the applications of computer graphics.

CVG was first proposed in Chen and Tucker [19], where various operations and their laws were given, the high-level representation of graphics objects using CVG terms explained, and recursive rendering via structural induction on terms introduced. In [19], the scalar fields are total functions, which simplifies the algebra. A fuller mathematical treatment of CVG, including approximation, is in Johnson [42].

Computation in CVG involves computation on real numbers, real-valued functions and operators. To understand the semantics of the CVG programming the framework needs to be analysed by a computabilty theory for topological spaces. In Blanck, Stoltenberg-Hansen and Tucker [16] we consider computability with partial functions and apply the theory to the computability of CVG algebras, such as the RGB algebras.

10.8 Analogue and digital systems

In computer science the interfaces between continuous and discrete data types are not well understood. Domains and topological spaces are designed to model continuous data, but they can also model discrete data. Can domain representations model computation with continuous and discrete data in a uniform way? Yes.

Consider analogue and digital data and the interface between them. A *data stream* is a sequence of data indexed by time. Mathematically, we model data streams by functions

$$s: T \to A$$

where $s(t)$ = datum or measurement from A at time t. The functions may be total or partial.

There are several cases of practical importance to consider, especially the purely digital case:

$$\text{discrete time } T = \mathbb{Z} \text{ and discrete data } A = \{0, 1\};$$

and the purely analogue case:

$$\text{continuous time } T = \mathbb{R} \text{ and continuous data } A = \mathbb{R}.$$

We model computation with these streams by mappings of the form

$$F: [T \to A] \to [T' \to B]$$

where T, T' are time scales and A, B are data types. The stream transformations include analogue-to-digital and digital-to-analogue transformations.

We have seen a number of mathematical tools to tackle the problem of analysing the semantics of analogue *versus* digital computing and signal processing, starting with domain representations of the reals. In applying domain representations and computability theory we focus on streams and stream transformers that are continuous functions. (The functions may be partial to help model discontinuities in streams.) The interface between analogue and digital computation is studied in [15], using domain representations of spaces with the compact-open topology.

10.9 Applications using continuous domains

Let us remind the reader that in this introduction to domain representation theory we have used algebraic domains exclusively and concentrated on our own interests. As emphasised earlier, one can use many types of ordered structure for representation. In particular, A. Edalat has used continuous domains to represent topological spaces in many applications, including several areas we have not discussed here.

The early applications of continuous domain representations focussed on semantic modelling of case studies of mathematical approximation, including iterative maps and integration, see Edalat [24, 25, 26]. This was done without emphasis on computability. A great deal of effort was devoted to using domains to develop software for exact arithmetic on computers.

With the rise of Computable Analysis, later studies of metric spaces in Edalat and Heckmann [27], real numbers in Edalat and Sünderhauf [31] and

Banach spaces in Edalat and Sünderhauf [32] looked at computability and may be compared with approaches based on algebraic domains mentioned above.

Recently, new subjects have been started. There is extensive work on computational geometry and Constructive Solid Geometry (CSG), which is a modelling technique well-established in CAD, see Edalat and Lieutier [28]. CSG is a precursor to CVG mentioned in subsection 10.7. Some first steps into the rich and vast subject of calculus and solving differential equations have been also taken in [29, 30].

The use of continuous domains has the advantage that often (but not always) D^R may be chosen as the set of maximal elements of D and the definition of representability is then reformulated in these terms. A disadvantage is that the theory of continuous domains is more involved. We have stuck to algebraic cpos and domains because of their simplicity and the fact that they arise from our consideration of approximation structures. Moreover, it is well known that every continuous cpo is a retract of an algebraic cpo. It follows that the two approaches of using continuous representations or merely algebraic representations are essentially equivalent.

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