Computations via experiments with kinematic systems[1]

E.J. Beggs[2] and J.V. Tucker[3]

University of Wales Swansea,
Singleton Park,
Swansea, SA3 2HN,
United Kingdom

Abstract

Consider the idea of computing functions using experiments with kinematic systems. We prove that for any set A of natural numbers there exists a 2-dimensional kinematic system B_A with a single particle P whose observable behaviour decides $n \in A$ for all $n \in \mathbb{N}$. The system is a bagatelle and can be designed to operate under (a) Newtonian mechanics or (b) Relativistic mechanics. The theorem proves that valid models of mechanical systems can compute *all possible* functions on discrete data. The proofs show how *any* information (coded by some A) can be embedded in the structure of a simple kinematic system and retrieved by simple observations of its behaviour. We reflect on this undesirable situation and argue that mechanics must be extended to include a formal theory for performing experiments, which includes the construction of systems. We conjecture that in such an extended mechanics the functions computed by experiments are precisely those computed by algorithms. We set these theorems and ideas in the context of the literature on the general problem "Is physical behaviour computable?" and state some open problems.

Keywords: foundations of computation; computable functions and sets; Newtonian kinematic systems; Relativistic kinematic systems; foundations of mechanics; theory of Gedanken experiments; non-computable physical systems.

1 Introduction

Consider the idea of computing functions by means of experiments with physical systems. Suppose each computation by a physical system is based on *running an experiment* with three stages:

(i) input data x are used to determine initial conditions of the physical system;

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[2]Department of Mathematics. Email: e.j.beggs@swansea.ac.uk

[3]Department of Computer Science. Email: j.v.tucker@swansea.ac.uk

(ii) the system operates for a finite time; and

(iii) output data y are obtained by measuring the observable behaviour of a system. The function f computed by a series of such experiments is simply the relation $y = f(x)$.

Typically, experiments on physical systems compute functions on continuous data, such as functions of the form $f\colon \mathbb{R}^n \to \mathbb{R}^m$, on the set $\mathbb{R}$ of real numbers; but they can also compute functions on discrete data, such as functions of the form $f\colon \mathbb{N}^n \to \mathbb{N}^m$, on the set $\mathbb{N}$ of natural numbers. The questions arise:

What are the functions computable by experiments with physical systems? How do they compare with the functions computable by algorithms?

This concept of *experimental computation* is both old and general. It can be found in ideas about (a) technologies for making machines and (b) modelling physical and biological systems. The concept is also complicated and in need of systematic theoretical investigation. In contrast, computability theory, founded by Church, Turing and Kleene in 1936, is a deep theory for the functions computable by algorithms on discrete data (Rogers [46], Odifreddi [36], Griffor [26], Stoltenberg-Hansen and Tucker [54]); it is being extended to continuous data (Aberth[1], Pour-El and Richards [44], Blum et al [8], Tucker and Zucker [60, 61], Weihrauch [62]).

Where there are instruments and machines for aiding calculation one can view a computation as an experiment with a physical system. Current technologies for computing and communication, such as those based on electronics, optics and quantum mechanics, involve the idea of experimental computation. With any new technology comes the question:

Can experimental computation by a system based on a given physical technology define less or more functions than computation by algorithms?

Conversely, where there are physical systems that can be initialised and whose behaviour is observable in some way, functions can be extracted from experiments and used to express their results. For different types of physical system, there have been attempts to pose and answer the question:

Does there exist a physical system of some given type that exhibits non-algorithmically computable behaviour?

We discuss attempts to pose and answer these questions in Section 6.

There is no shortage of examples, results, discussion and debate on experimental computation in special situations. For example, there are a number of ways to simulate a Turing machine by physical systems (e.g., billiard balls) or classes of dynamical systems (e.g., cellular automata). However, the questions about non-computability above do not yet have *definitive* answers. There is plenty of speculative discussion. Some examples of non-computability are incomplete and, strictly speaking, have the status of conjectures. Some theorems encode non-computability in general classes of mathematical systems (e.g., ODEs) rather than models of specific physical systems (e.g., pendula). There can be problems, too, with the validity of the algorithmic model defining computability in the case of continuous data. We discuss this in Section 6.

For definitive answers, particular examples need to be studied and the precise *physical* concepts and laws identified that permit or prevent non-computable functions and behaviours. Furthermore, a conceptual analysis is needed to formulate systems of axioms that characterise abstractly, and in general, the information processing capabilities of physical systems.

Here we will examine some idealised experiments with idealised physical systems. An idealised physical system is a system whose specification and behaviour is governed by an appropriate set of physical laws. We will show there exist simple kinematic systems, which operate under the theories of Newtonian and Relativistic mechanics, that can decide the membership of *any* subset A of the set $\mathbb{N} = \{0, 1, 2, \ldots\}$ of natural numbers. The systems are infinite bagatelles that are based on simple energy and momentum conservation principles. They each require unbounded space, time and energy to decide $n \in \mathbb{N}$ for all n. The Newtonian case is simple. The relativistic case might be considered to be more realistic and it also has a useful theoretical property, a *maximum propagation speed* for objects or information, the speed of light c. Instead of unbounded velocity in the Newtonian case, in the relativistic case we exploit the fact that the mass of a particle is unbounded as its speed approaches c.

Theorem 1.1. *Let $A \subseteq \mathbb{N}$. There exists a 2-dimensional kinematic system with a single particle P whose observable behaviour decides A. More specifically, the system is an infinite bagatelle for which the following are equivalent: given any $n \in \mathbb{N}$*

(i) $n \in A$

(ii) In an experiment, given initial velocity V_n the particle P leaves and returns to the origin within a known time T_n.

The system can be designed to operate under

(a) Newtonian mechanics or

(b) Relativistic mechanics.

The velocity V_n and the time T_n are easily calculated from n and so by simply projecting the particle and watching the clock while waiting for its return, we can decide A. Thus, mechanical systems *exist* to compute by experiment *all* the functions on $\mathbb{N}$. This fact suggests that the elementary theory of kinematics is undesirably strong. For example, it suggests that any conceivable discrete information can be represented in the behaviour of a ball rolling in along a line.

Now, the proofs explore the coding of non-computable sets into the structure of kinematic systems, rather than into their operation. The simple physical laws that govern their observation and operation will allow any experiment on any given bagatelle. However, it is through the *description* of the system that the computation of any A is possible. If the analysis of the experiment concerned not just the observation of an existing system but the process of *assembly* or *construction* of the bagatelle then further conditions on the system would be needed. We suggest a form for such an analysis that would restrict the subsets of $\mathbb{N}$. Thus, the bagatelles show that a formal account of experimentation, that includes the specification and construction of mechanical systems, is needed to answer the questions above. This critique is the subject of Section 5.

In the case of the bagatelle there are certain natural assumptions on experiments that would allow them to compute only the semicomputable and computable subsets of $\mathbb{N}$. Indeed, by choosing $A \subseteq \mathbb{N}$ to be a complete semicomputable set then the construction yields a new universal computer:

Corollary 1.2. *There exists a 2-dimensional kinematic system with a single particle P*

that is a universal machine for the the computable partial functions on $\mathbb{N}$*, i.e. the bagatelle computes by experiment all and only the computable partial functions on* $\mathbb{N}$*.*

The structure of the paper is this. In Section 2 we describe the construction of a general type of infinite bagatelle. In Section 3 we complete the description of a bagatelle that decides the membership relation for A under Newtonian mechanics, and in Section 4 we re-design the bagatelle to decide the membership relation for A under Relatvistic mechanics. In Section 5 we reflect on the examples and argue that mechanics is in need of a formal theory of experimentation to answer the questions. Finally, in 6 we discuss earlier work on these questions, a programme for their systematic investigation, and some open problems for kinematics.

2 Experiments with an infinite bagatelle

We describe the structure of our bagatelle, and the steps involved in using the bagatelle to compute. The structural form and the experimental procedure of the bagatelle is common to both the Newtonian and Relativistic machines.

Experiments with the bagatelle We consider a bagatelle game. A ball is fired into the bagatelle machine with a specified velocity, and the ball may or may not return in a given time period. Nothing else about the bagatelle is externally observable. The instructions for operating the bagatelle consist of a list of velocities V_1, V_2, V_3, etc. and a list of times T_1, T_2, T_3, etc. These numbers are precisely the same for all Newtonian machines. Similarly the lists of velocities and times are uniform for all relativistic machines.

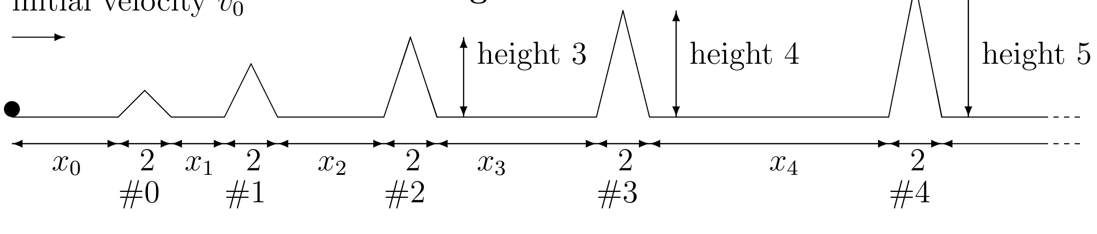
Each machine can define a subset A of the natural numbers $\mathbb{N}$ as follows: Given $n \in \mathbb{N}$, you fire a ball into the machine at initial velocity V_n, and the ball returns in a time $\mathrm{Return}(V_n)$. Then

$$\begin{array}{lll} n \in A & \text{if and only if} & \mathrm{Return}(V_n) \leq T_n \ , \\ n \notin A & \text{if and only if} & \mathrm{Return}(V_n) \geq T_n + 1 \ . \end{array} \tag{1}$$

The gap between T_n and $T_n + 1$ ensures that we only have to ensure measurement of time to a certain accuracy. Also note that the result can be determined in a finite time $T_n + 1$, even though the ball might never return.

Structure of the bagatelle If we were to lift the lid on the bagatelle, we would see something like this:

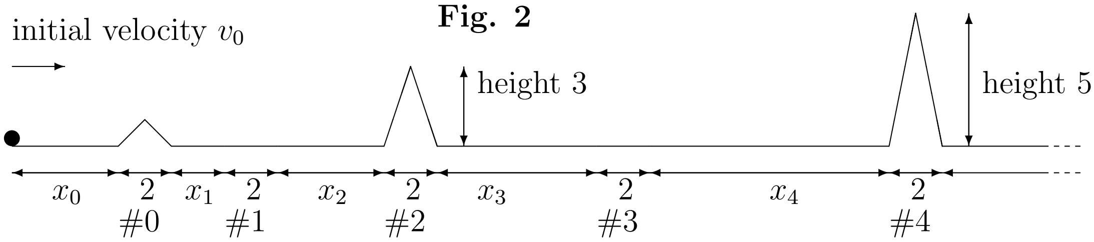
Fig. 1



The machine continues indefinitely off the right hand side. At time $t = 0$ the ball starts from position $x = 0$ with initial velocity v_0. It then crosses, or fails to cross, potential barriers placed in the way along the x-axis. For integer $n \geq 0$ the barrier #n has height $n+1$ and width 2. For simplicity we assume that it is has the shape of an isoceles triangle. The reader who is anxious about the sharp corners should compute the arbitarily small corrections in the formulae given by introducing arbitrarily small smoothings of the corners. There is a flat gap (at height 0) between $\#n$ and $\#n+1$ of length x_{n+1}. We will give the value of the numbers x_n later.

To specify the internal workings of a bagatelle we need a subset A of $\mathbb{N}$. The bagatelle has a potential barrier of height $n+1$ at position $\#n$ if $n \in A$, and a flat track if $n \notin A$. For example, the subset of even natural numbers would correspond to a machine looking like figure 2:

Fig. 2



The reader should note that we suppose that there is no friction or external force acting on the ball. We also assume that the ball is not spinning (or at least that, if it is spinning, that its moment of inertia is zero).

Operation of the bagatelle When a ball hits a potential barrier of height H at velocity v_0, there are three possibilities:

1) It has sufficient energy to cross the barrier, and crosses it in time $C(v_0, H)$ from one base to the other. We assume that $C(v_0, H) \geq 2/v_0$, i.e. that it takes the ball at least as much time to cross the barrier as to travel on a flat track if there is no barrier.

2) It has insufficient energy to cross the barrier, and rolls up and back down in time $B(v_0, H)$ from base to base.

3) It has exactly the right amount of energy to reach the top. We shall take care to avoid this case, as it gives rise to discontinuities in the return time, and the behaviour is critically dependent on the shape of the top of the barrier.

Take V_n to be an initial velocity which ensures that the ball has enough energy to cross all barriers $\#j$ for $j < n$, but that the ball will not cross, but roll back down from $\#n$. Suppose the ball is fired at this velocity on the bagatelle specified by the subset A.

If $n \in A$, then the time of return to the initial point would be

$$\text{Return}(V_n, A) \;=\; \frac{2}{V_n}\Big(\sum_{j\leq n} x_j + \sum_{j<n,\ j\notin A} 2\Big) \;+\; 2\sum_{j<n,\ j\in A} C(V_n, j+1) \;+\; B(V_n, n+1)\;. \quad (2)$$

The first term is given by the ball traversing the flat track at height zero, and the second by the ball crossing over the barriers of height less than n. Remember that both these are done twice, once in either direction. The last term is the time taken for the ball to be reflected from the barrier $\#n$.

However if $n \notin A$, the time of return would be

$$\text{Return}(V_n, A) \;\geq\; \frac{2}{V_n}\Big(\sum_{j\leq n} x_j + \sum_{j<n,\ j\notin A} 2\Big) \;+\; 2\sum_{j<n,\ j\in A} C(V_n, j+1) \;+\; \frac{2\,x_{n+1}}{V_n}\;. \quad (3)$$

This time is based on the fact that if the ball did return, it would have to travel twice over a flat track of length x_{n+1}. Of course the ball might never return, as there might be no more barriers for it to cross, but this case is included in the inequality.

Choice of the displacements x_n We want an experiment to determine if $n \in A$, and do not want the result confused by other elements of A. However our results (2) and (3) depend on elements in A which are less than n. We deal with this by considering the values taken as we vary A, and choose x_n and T_n to be independent of A: First we choose the sequence $x_n \geq 0$ satisfying the inequalities

$$x_{n+1} \geq \sum_{j<n} \Big(V_n \, C(V_n, j+1) - 2 \Big) + \frac{V_n(B(V_n, n+1) + 1)}{2} . \tag{4}$$

Definition of the time bounds T_n Then we set T_n by

$$T_n = \frac{2}{V_n} \sum_{j \leq n} x_j + 2 \sum_{j<n} C(V_n, j+1) + B(V_n, n+1) . \tag{5}$$

If $n \in A$, remembering that $C(v_0, H) \geq 2/v_0$ we have from (2):

$$\text{Return}(V_n, A) \leq \frac{2}{V_n} \sum_{j \leq n} x_j + 2 \sum_{j<n} C(V_n, j+1) + B(V_n, n+1) = T_n. \tag{6}$$

Correspondingly for $n \notin A$, from (3) we have

$$\text{Return}(V_n, A) \geq \frac{2}{V_n} \Big(\sum_{j \leq n} x_j + \sum_{j<n} 2 \Big) + \frac{2\, x_{n+1}}{V_n} \geq T_n + 1 . \tag{7}$$

It remains to find formulae for V_n, B and C in the Newtonian and relativistic cases.

3 Newtonian kinematics

The initial kinetic energy of the ball of mass m with any initial velocity v_0 is $\frac{1}{2} m v_0^2$. The potential energy of the ball at height h above the initial point is mgh, where g is the acceleration due to gravity (on the Earth's surface, this is about $9{\cdot}8$ meters/second^2). The principle of conservation of energy then gives the velocity v of the ball at a height h using $\frac{1}{2} m v_0^2 = \frac{1}{2} m v^2 + mgh$. It follows that the maximum height H that the ball can attain is given by $\frac{1}{2} m v_0^2 = mgH$, i.e. $H = \frac{1}{2} v_0^2 / g$. We set V_n to be the initial velocity for which the maximum attainable height is $n + \frac{1}{2}$, i.e.

$$V_n = \sqrt{g(2n+1)} . \tag{8}$$

Proposition 3.1. *The time taken for a ball with initial velocity v_0 to climb a slope of gradient n to a height h (less than the maximum height $\frac{1}{2} v_0^2 / g$) is*

$$\frac{v_0 - \sqrt{v_0^2 - 2gh}}{g} \sqrt{1 + \frac{1}{n^2}} .$$

Proof. We start the slope at the point $(x, y) = (0, 0)$, so the equation of the slope is $y = nx$. On rearranging the conservation of energy equation, we see that at height y the particle has velocity $v = \sqrt{v_0^2 - 2gy}$. The length of slope from height y to $y + dy$ is given by Pythagoras' theorem as $\sqrt{(dx)^2 + (dy)^2}$, or using the equation $y = nx$, as $dy \sqrt{1 + n^2}/n$. The time taken to move from height y to $y + dy$ is the distance divided by the velocity, or $dy \sqrt{1 + n^2}/(nv)$. This gives the total time to climb to height h as the integral

$$\int_{y=0}^{h} \frac{dy \sqrt{1 + n^2}}{n \sqrt{v_0^2 - 2gy}} = \frac{v_0 - \sqrt{v_0^2 - 2gh}}{g} \sqrt{1 + \frac{1}{n^2}} \ . \quad \square$$

Corollary 3.2. *The time taken for a ball with initial velocity v_0 to climb a slope of gradient n to its maximum attainable height is*

$$\frac{v_0}{g} \sqrt{1 + \frac{1}{n^2}} \ .$$

Corollary 3.3. *Using the definition of V_n in (8), we have, for $j \le n$,*

$$\begin{aligned} C(V_n, j) &= 2 \frac{\sqrt{2n+1} - \sqrt{2n - 2j + 1}}{\sqrt{g}} \sqrt{1 + \frac{1}{j^2}} \ , \\ B(V_n, n+1) &= 2 \frac{\sqrt{2n+1}}{\sqrt{g}} \sqrt{1 + \frac{1}{(n+1)^2}} \ . \end{aligned}$$

Proof. We use the formulae given in 3.1 and 3.2, remembering that it takes the same time to roll down as to climb up. $\square$

Remark 3.4. *Here we calculate asymptotic bounds on the time taken by the Newtonian bagatelle to decide if $n \in A$ or not. From (8) and 3.3 we see that V_n, $C(V_n, j)$ and $B(V_n, n + 1)$ are all $O(\sqrt{n})$. From (4) we can choose x_n to be $O(n^2)$, and from (5) we have T_n to be $O(n^{5/2})$.*

4 Relativistic kinematics

The relativistic mass of a ball of rest mass m travelling at velocity v is $M = m/\sqrt{1 - v^2/c^2}$, where c is the speed of light. The momentum of the ball is Mv, and we use the usual formula that force is the rate of change of momentum. On a slope inclined at an angle α to the horizontal, we have $\frac{d}{dt}(Mv) = -Mg \sin(\alpha)$. On rearranging and differentiating this yields $\frac{dv}{dt} = -g(c^2 - v^2) \sin(\alpha)/c^2$. On integrating we get

$$v = c \tanh(g(b - t) \sin(\alpha)/c) \ , \tag{9}$$

where b is a constant. The initial velocity is

$$v_0 = c \tanh(gb \sin(\alpha)/c) \ , \tag{10}$$

which, using a hyperbolic trig identity, becomes the useful formula

$$\cosh(gb \sin(\alpha)/c) = 1/\sqrt{1 - v_0^2/c^2} \ . \tag{11}$$

The distance travelled along the slope as a function of time is given by integrating (9)

$$d = \frac{c^2}{g\,\sin(\alpha)}\,\log\Big(\frac{\cosh(bg\sin(\alpha)/c)}{\cosh((b-t)g\sin(\alpha)/c)}\Big) ,$$

so the height as a function of time is

$$h = \frac{c^2}{g}\,\log\Big(\frac{\cosh(bg\sin(\alpha)/c)}{\cosh((b-t)g\sin(\alpha)/c)}\Big) . \tag{12}$$

The maximum height achieveable occurs when $t = b$, and is

$$h_{max} = \frac{c^2}{g}\,\log\big(\cosh(bg\sin(\alpha)/c)\big) . \tag{13}$$

If the maximum height is set to $n+\frac{1}{2}$, then using (11) and (13) the corresponding initial velocity V_n is given by

$$V_n = c\sqrt{1-e^{-(2n+1)g/c^2}} . \tag{14}$$

Proposition 4.1. *The time taken for a ball with initial velocity* v_0 *to climb a slope of gradient* $\sin\alpha$ *to a height* h *(less than the maximum height) is*

$$\frac{c}{g\,\sin\alpha}\Big(\tanh^{-1}\Big(\frac{v_0}{c}\Big) - \cosh^{-1}\Big(\frac{e^{-gh/c^2}}{\sqrt{1-v_0^2/c^2}}\Big)\Big) .$$

Proof. If we rearrange (12) we get

$$\cosh((b-t)g\sin(\alpha)/c) = \cosh(bg\sin(\alpha)/c)\,e^{-gh/c^2} ,$$

so we get t as

$$t = b - \frac{c}{g\,\sin\alpha}\,\cosh^{-1}\Big(\cosh(bg\sin(\alpha)/c)\,e^{-gh/c^2}\Big) . \qquad \square$$

Corollary 4.2. *The time taken for a ball with initial velocity* v_0 *to climb a slope of gradient* $\sin\alpha$ *to its maximum attainable height is*

$$\frac{c}{g\,\sin\alpha}\,\tanh^{-1}\Big(\frac{v_0}{c}\Big) .$$

Corollary 4.3. *Using the definition of* V_n *in (14), we have, for* $j \le n$,

$$C(V_n, j) = \frac{2\,c\,\sqrt{1+j^2}}{g\,j}\Big(\cosh^{-1}\big(e^{(2n+1)g/(2c^2)}\big) - \cosh^{-1}\big(e^{(2n+1-2j)g/(2c^2)}\big)\Big) ,$$

$$B(V_n, n+1) = \frac{2\,c\,\sqrt{1+(n+1)^2}}{g\,(n+1)}\,\cosh^{-1}\big(e^{(2n+1)g/(2c^2)}\big) .$$

Proof. We use 4.1 and 4.2, with (11) and (14) supplying the formula

$$\cosh(bg\sin(\alpha)/c) = e^{(2n+1)g/(2c^2)} . \qquad \square$$

Remark 4.4. *Here we calculate asymptotic bounds on the time taken by the relativistic bagatelle to decide if* $n \in A$ *or not. From 4.3 we see that* $C(V_n, j)$ *and* $B(V_n, n+1)$ *are both* $O(n)$. *For* n *large,* $V_n \cong c$. *From (4) we can choose* x_n *to be* $O(n^2)$, *and from (5) we have* T_n *to be* $O(n^3)$.

5 Commentary on the Bagatelle

5.1 Corollaries

Corollary 5.1. *Any function* $f : \mathbb{N} \to \mathbb{N}$ *can be computed by a Newtonian or Relativistic bagatelle*

Proof. Let G_f be the graph of f. Choose an injective function $c : \mathbb{N}^2 \to \mathbb{N}$ such as $(x, y) \mapsto 2^x.3^y$ and code the graph G_f as the set $c(G_f)$. A bagatelle B_A based on $A = c(G_f)$ would enable f to be computed experimentally by the mechanical system. □

Corollary 5.2. *There exist Newtonian and Relativistic bagatelles that are universal machines for the computable partial functions on* $\mathbb{N}$*, i.e. the bagatelles compute by experiment all and only the computable partial functions on* $\mathbb{N}$.

Proof. Choose a bagatelle B_A based on $A = c(G_U)$, the coded graph of a universal partial recursive function U. This would enable U to be computed experimentally by the mechanical system. □

5.2 Interpretations

All the bagatelles are systems that are valid in theoretical mechanics. Clearly, the structure of the bagatelle B_A is based on the set A and there is nothing in mechanics that prevents or cautions us from defining such systems for *any* set A; thus, the bagatelle B_A is a legal mechanical system.

Now, *given any* bagatelle B_A then all the experiments needed to decide $n \in A$ can be carried out using the following *primitive experimental actions*:

(i) project a particle with arbitrary large energy (for arbitrary large natural numbers);
(ii) observe a fixed point in space;
(iii) measure arbitrarily large times on a clock; and
(iv) calculate with simple algebraic formulae.

Indeed, the actions required are very simple and uncontentious. Thus, we have the extreme and worrying result that valid or legal Newtonian and relativistic systems exist to compute any set or function on $\mathbb{N}$.

We call this the *classical interpretation* of the theorems because this is the standard way of interpreting theorems in classical mechanics. In particular, the existence of the bagatelle is proved using classical reasoning.

However, suppose the account of the experiment is required to explain how the mechanical system is *constructed*, as well as what primitive experimental actions are needed to set initial states and observe behaviour. Then we find we have an interesting problem.

What assumptions underly our idea of an experiment with the bagatelle?

Extending the informal ideas of Geroch and Hartle [22] on experiments designed to measure quantities (see Section 6.2), then the experiments needed involve primitive experimental actions of the following kind:

(i) selection steps from a source of unlimited natural resources;
(ii) primitive construction steps (e.g., make a barrier and place a barrier);
(iii) primitive experimental steps (e.g., project the particle, measure time);

(iv) schedule the three kinds of primitive steps, which may be interleaved, according to a global laboratory clock

With these actions we can postulate a precise form for an experiment:

"Definition" An *experiment* is a finite or infinite process made of primitive construction or experimental steps indexed by the laboratory clock.

Now, set against this definition of an experiment, one problem is that the sequence of primitive steps in the construction of the system B_A will involve knowledge of the set A - probably precisely the knowledge the system B_A is being designed to reveal, making the purpose of the experiment redundant. Since we are interested in the nature and use of mechanical systems, this point about redundancy is not so interesting. What conditions will be required on A to allow experiments on B_A that are valid in this extended sense? An experiment could run as follows.

Suppose A is given by some increasing enumeration $A_0, A_1, A_2, \ldots$ of finite subsets with for each $i \in \mathbb{N}$, $A_i \subset A_{i+1}$ and $A = \bigcup_{i \in \mathbb{N}} A_i$. Suppose that A_i has i elements of A. Then to make an experiment to decide if $n \in A$ then we need an experimental procedure to construct a finite part of the bagatelle. This finite part will have the form B_{A_k} for some $A_k \subset A$. It will have k potential barriers located by the k elements of A_k.

An indepenedent laboratory clock will schedule the construction of the approximating bagatelle B_{A_k} and an experiment to decide $n \in A_k$. If the experiment confirms that $n \in A_k$ then we know that $n \in A$.

However, if the experiment confirms that $n \notin A_k$ then we do not know that $n \notin A$. This result can change as k increases and more and more elements of A appear and the bagatelle grows. Each negative result must be repeated and so the experiment becomes a *search* for a positive result, secure in the knowledge that if $n \in A$ then an experiment with some part B_{A_k} of the bagatelle B_A will find it.

Thus, when we include the construction of the bagatelle in the primitive steps we have a proof that A is decidable by experiment if, and only if, finite subsets of A can be generated by experiment. Indeed, we are close to a proof that A is decidable by experiment if, and only if, A is recursively enumerable subset of $\mathbb{N}$.

We call this the *constructive interpretation* of the theorems because we are adding principles of system construction to experimental principles of classical mechanics. In particular, the existence of the bagatelle is here proved using constructive reasoning.

6 Computable and non-computable physical systems

The general questions on computing with physical systems posed in the Introduction, and even the special cases for particular kinds of physical system, are difficult problems. To answer them, physical theories must be combined with computability theories, and a clear account of the conduct of idealised experiments is necessary. Gedanken experiments have be used since Galileo and are a complex philosophical subject in their own right, of course (see, e.g., Brown [12], Bohr [9], Koyre [27], Kuhn [30]). To cite an example in kinematics, gedanken experiments related to Zeno's paradoxes have re-surfaced in philosopical debates about infinite machines and Newtonian supertasks (Perez Laraudogoitia [37, 38] and Alper and Bridger [3]).

Attempts to answer the questions often involve computable functions on continuous data. Computation on continuous data can use algorithms that approximate infinite data and so the concept consists of three ideas:

$$\text{Computation} = \text{Data} + \text{Programs} + \text{Approximation}.$$

There are many ways to model *each* of these three ideas. For example, data can be abstractly specified or concretely represented, programs can made from many different constructs, and approximation can be expressed via orderings, norms, metrics and topologies. We do not yet possess a well understood theory of algorithmic computation for infinite data, even on $\mathbb{R}^n$. However, in the main computability theories on infinite data one finds that if f is computable then f maps computable data to computable data. Therefore, in the search for non-computability, it is common to seek systems that define a function f by experiment such that f returns non-computable output from computable input, since such an f cannot be computable.

With these difficulties in mind, we consider some different approaches to the problems by surveying representative work. This provides a landscape against which to appreciate the study of mechanical examples of the kind given here.

6.1 The search for non-computability

The question "Is physical behaviour computable?" was asked in computability theory in, e.g., Kreisel [28]. The problem is unresolved, it will not go away, and has become more confusing, difficult and fascinating (Cooper and Odifreddi [14]).

Wave mechanics A major attempt at an answer was by Pour El and Richards, who proposed "No". In Pour El and Richards [42] they showed that there are solutions of the 3-dimensional wave equation with computable initial values that are are not computable over unit time [0, 1]. The notion of computable was based on the uniform norm on $C[\mathbb{R}^3, \mathbb{R}]$. The mathematical fact was later analysed in terms of the computability of operators on Banach spaces in Pour El and Richards [44], and in the general setting of partial homomorphisms of arbitrary metric partial algebras in Stoltenberg-Hansen and Tucker [57]. Because the wave equation is a fundamental model of physical phenomena, Pour El and Richards suggested that the result indicated that there exist physical systems that could show non-computable behaviour. However, the experimental basis of the proposal was too weak to support the suggestion, as pointed out in Kreisel [29].

Recently, Weihrauch and Zhong [65] have re-visited the wave equation, claiming that by using the appropriate norms, solving the wave equation is a computable problem. They have shown that using the C^1 norm on the differentiable functions $C^1[\mathbb{R}^3, \mathbb{R}]$ (with uniform convergence of both the functions and their partial derivatives) and the uniform norm on the continuous functions $C[\mathbb{R}^3, \mathbb{R}]$, that the wave equation solution operator $S' : C^1[\mathbb{R}^3, \mathbb{R}] \times C[\mathbb{R}^3, \mathbb{R}] \times \mathbb{R} \to C[\mathbb{R}^3, \mathbb{R}]$ is computable. Here $S'(f, g, t)$ is the solution to the wave equation at time t which takes the value f at time zero and has velocity g at time zero. If the same type of norms are desired in both the initial and final conditions, they also showed that for all real numbers s and computable times t that $S'(t) : H^s[\mathbb{R}^3, \mathbb{R}] \times H^{s-1}[\mathbb{R}^3, \mathbb{R}] \to H^s[\mathbb{R}^3, \mathbb{R}] \times H^{s-1}[\mathbb{R}^3, \mathbb{R}]$ was computable, where H^s is a Sobolev space of functions. Hence they propose that the answer is, in fact, "Yes" in the case of n-dimensional wave systems. In Weihrauch and Zhong [66] is a proof that the Schrödinger equation has computable solutions.

Let us try to convert the Pour El and Richards method to a 'real' experiment. We start with a flat sheet of ice, and use a computer controlled tool to shape the surface. Then we instantaneously melt the ice, and the wave equation takes over. After a certain time we make a measurement of the wave, and a 'noncomputable' result emerges. The tool shapes the surface according to a computable function, whose derivative is not computable (Myhill [35]). By a computable function, we mean that its value at a given point can be calculated to a given precision given enough time (i.e. clock cycles). If we insist that the construction be performed in a finite number of clock cycles, we could shape the surface to, say, one micron of the theoretical function. But the uniform convergence of functions does not imply convergence of the derivatives, as previously noted. In other words we cannot ensure that the derivative is anything like the theoretical value in a finite time, so the result of any such experiment is likely to deviate widely from the calculated (noncomputable) result. The only way to achieve the theoretical result would be to have a 'deus ex machina' give the experimenters a precisely shaped ice sheet to begin with. This is just the situation which occurs with the bagatelle. To make one from a sheet of metal with a computer controlled tool, we either settle for a computable set of barriers, or we have to wait an infinite amount of time for the construction (and for many subsets even an infinite amount of time will not do). In other words, the bagatelle throws up the same philosophical points as the Pour El and Richards method, but does so in a more obvious fashion.

Physical simulations of Turing machines To define a physical system that can simulate a Turing machine one has to define the system, describe its operation and show how an experiment with the system mimics the behaviour of a Turing machine. A sound argument should lead to the claim that the system can realise or implement all computable behaviour and is a technology for digital computation. It *may* also yield results about non-computable behaviour of the physical system from the undecidability of the halting problem for Turing machines.

A good example of this approach is Moore [33] on the "unpredictability" of physical systems. Moore shows how to model a Turing machine by a shift map, and, in turn, suggests a kinematic system, with a single particle guided by pin-ball "mirrors" in a 3D potential, that is "equivalent" to a Turing machine. He argues that a Gedanken experiment with a system corresponding with a universal Turing machine would have undecidable behaviour. The motivation of the investigation is to show that such a system is far more unpredicable than the common or garden chaoic system. Moore's arguments are suggestive rather than rigorous so, strictly speaking, the result is a conjecture.

Implementing Moore's system in some idealised mechanics is quite interesting. To model a Turing machine, the successive reflections have to be done with complete accuracy, any deviation will be magnified by successive reflections. In our idealised world we may assume that light has no discernable wave nature, and will not diffract when passing through the system. However we also have to assume that the mirrors are perfectly flat and perfectly positioned, which means that they cannot be made of atoms as we know them. Such a rejection of the atomic hypothesis can easily lead to other strange constructions, such as successive mechanical models of Turing machines, each half the size of the last, and each doing its calculation in half the time of the last. If these were connected to perform successive steps of a calculation, infinitely many clock cycles could be performed in finite time.

Suggestions for simulating digital logic by kinematics are in Fredkin and Toffoli [20]

Analogue computers Analogue computation as conceived by Lord Kelvin [58], V Bush [13], and D Hartree [24], is experimental computation. The functions are of the form $f\colon \mathbb{R}^n \to \mathbb{R}^m$ and the physical systems are made from mechanical or electro-mechanical components. The theory of analogue computers is modest. A general purpose analog computer (GPAC) was introduced in Shannon [48] as a model of the Differential Analyser of Bush [13]. Shannon discovered that a function can be generated by a GPAC if, and only if, it is differentially algebraic, but his proof was incomplete. An analysis in Pour El [40] yielded a new stronger model and a new proof of the equivalence (and some new gaps corrected in Lipshitz and Rubel [31]). Using the characterisation in terms of algebraic differential equations, these analogue models were shown not to compute all computable functions on $\mathbb{R}$ (Pour El [40]). These models are close to the practice of analogue computing until the 1960s. That an undecidable predicate of a computable function on $\mathbb{R}$ might be experimentally computable by a suitable analogue machine was observed in Scarpellini [47].

Recently, the theory of analogue computing has been restarted by C Moore with very general mathematical models (Moore [34]). These models define functions by schemes rather like Kleene's, but with primitive recursion replaced by integration and others added, but can define functions beyond the class of computable functions on $\mathbb{R}$. In Graca and Costa [23] another model close to the GPAC has been shown to be equivalent with a subclass of Moore's functions (those defined by composition and integration).

Neural networks Among the first attempts to model physical systems as computing devices is McCulloch and Pitts logical models of networks of neurones. Neural networks have been influential in digital computing (e.g., von Neumann's abstraction of computer architecture, Kleene's regular expressions, parallelism). They have also involved the interface between discrete and continuous notions.

The theory of neural networks is vast. The McCulloch and Pitts proposal that neural tissue can be modelled as hybrid logical/algorithmic networks has led to many results that confirm that the answer to the question "Are neural systems computable?" is "Yes" (e.g., see Holden et al [25]). However, a convenient survey of models and a proposal that some hybrid nets are not is in Siegelmann [52]. From the point of view of a theory of experimental computation, the arguments for this negative answer are not adequate, as the analysis in Davis [17] demonstrates. The strong debate of Penrose's proposals in [39] is also a rejection of the positive answer.

Quantum computing The experimental nature of computation, which our kinematic computers illuminate, is also the basis of the more complex field of quantum computation. Informal notions of quantum algorithms, computers and circuits have been developed and the the physical aspects of the Church-Turing Thesis discussed since Benioff [6], Deutsch [18, 19] and Yao [67]. Comparisons with classical computation have been focussed on the superior speed of quantum computation. An early rigorous definition of a quantum Turing machine is in Bernstein and Vazirani [7] where it is shown that the quantum

Turing machine can be simulated by a classical one and vice versa. However, the formulation of some quantum computer models rely on classical computability theory since they are require certain real number parameters must be computable. For example, in [7] probability amplitudes associated with state transitions are assumed computable or even rational; this hypothesis is relaxed in Adleman, DeMarrais, and Huang [2]. In some cases, quantum models are presented as a computable family of circuits. The justification of such extra hypotheses in experimental terms is an interesting problem. Many models for quantum computation are being developed and our understanding of this complex notion of computation is at an early stage. It seems not to be known if quantum cellular automata (Margolus [32]) can be simulated by a classical Turing machine.

Classical versus quantum systems Penrose's study and reflections on computability, physical laws and consciousness have stimulated a great deal of thought about the computability of physical systems and the mind (Penrose [39]). Relevant here is his conjecture that nature can produce non-computable processes that we can use but not at the level of classical physics. In da Costa and Doria [16] this idea is formulated as *Penrose's Thesis* and a "counter-example" suggested, based on da Costa and Doria [15], that shows classical mechanics can produce non-computable behaviour. We consider the counter-example is not convincing, not least because it complicated and fails to allow a robust experiment. Our bagatelle shows that the simplest examples of classical mechanics certainly allow non-computable behaviour but it is the notion of experiment that determines what can or cannot be harnessed.

Noncomputablility in dynamical systems There are several general classes of mathematical system that have their origins as classes of physical model, and about which computability results have been proved. These results simulate models such as Turing machines by ODEs or cellular automata, or encode undecidable problems in decison problems for dynamical systems. They are interesting because they suggest avenues for devising new idealised experiments with physical systems and are destined to belong to a theory framework. However, as mathematical systems, they have abstracted away from how physical ideas can be use to perform computation. Some examples follow. The existence of an ODE with computable initial conditions and no computable solution is proved in Pour El and Richards [41]. The simulation of machines by ODEs has been shown for finite automata in Brockett [11], and for Turing machines in Branicky [10]. Decision problems for the differential equations of mechanics are not new. An early problem in the qualititaive theory of celestial mechanics is *Poincare's Centre Problem* (see, e.g., Seigel and Moser [51]). A recent example of undecidability is da Costa and Doria [15] on the integration of Hamiltonians using quadratures, proved using the undecidability of the integration of elementary functions (Richardson [45]).

6.2 The search for a conceptual analysis

The search for non-computable aspects of particular physical examples, or of whole classes of mathematical models, must be complemented by a search for the general concepts and principles that enable physical systems to be used for computing. The aim is to find concepts, axioms and laws that can (a) embrace diverse examples of physical systems that may be said to compute; (b) explore the border between computability and non-computability; and (c) facilitate comparisons with general classes of mathematical models of physical systems and computers.

First, we might isolate the essential properties of

(i) experimentation, which are focussed on obtaining input and output, and

(ii) behaviour, which are focussed by the operation of a system.

We have sketched some simple ideas about experimentation in Section 5, enough to set up and criticise our bagatelles. A fuller analysis of experimentation would study the processes of observation, measurement and construction of a system, which are interconnected *and* dependent on some underlying theory for the system. In Geroch and Hartle [22], there is an attempt at characterising the numerical quantities that are measurable by experiment. We extended their conditions for experiments in Section 5. They argue that *any number computable by algorithms is measurable by experiment* because the process of applying an algorithm qualifies as an idealised experiment. Conversely, they argue that these measurable quantities are also computable numbers, at least when experiments are based on "conventional" physical theories. They ask if quantum gravity is a theory where this may fail. Although stimulating, their ideas about experiments fall short of axiomatic analysis.

A formal general property of experiments is the *continuity* of the input-output relation. The idea being that in performing meaningful experiments the results must display some robustness when they are repeated with small changes in initial and conditions and observable behaviour. This property is proposed in Kreisel [28], where the idea is refered to as *Hadamard's Principle* for well-posed systems. Continuity is also the key idea in the general mathematical arguments of Weihrauch and Zhong [65]. If continuity is a necessary characteristic of the function computed by an experiment then it is worth noting that on data types with metric space structures *Ceitin's Theorem* says, roughly, that *computablity implies continuity.* Studies of generalisations and converses of this theorem are Spreen [53] and Stoltenberg-Hansen and Tucker [57].

Thus, in designing physical systems for computation, one proviso is to avoid singular cases which give rise to discontinuities. In the mechanics of point particles, in the case of scattering off a barrier with a sharp corner, there is a singular case when a particle hits the vertex of the angle of the corner. In the case of the gravitational dynamics of point particles, a singular case arises when point particles collide. In the bagatelle we must choose input velocities that avoid the singular behaviour of a ball coming to rest at the top of a potential barrier.

However, to answer the questions, we seek sets of basic axioms that such an idealised physical system might satisfy if we are to use it in idealised experiments for computation. Our idea is to restrict attention to axiomatising the information processing capabilities of *systems in which the behaviour of the physical system is based on a "finite" transformation, propagation and observation of material, energy, or information.* In Beggs and Tucker [4] we give axioms for the local structures of systems and their local states.

An attempt at such an axiomatistion of machines for digital processing is Gandy [21] in which notions of space and causality are modelled using hereditarily finite sets. The conceptual analysis is frustratingly difficult to use and has been studied in some depth by Sieg [49, 50]. This analysis, although focussed on refining the idea of mechanical computation as portayed in Turing machines, is relevant to our problem. Digital computation by machines is also an example of physical computation and should properly fall within the scope of the problem. Digital computation is based on software and hardware systems that must be described *both* by abstract programs and machine architectures obeying the laws of logic, and by systems obeying the laws of physics. Our own axiomatisation owes something to Gandy's, though it is shaped by the study of computing with the systems of classical mechanics.

6.3 Concluding remarks on kinematic systems

In conclusion, experimental computation is not well understood and the questions asked in the introduction are open, even in the case of kinematics, possibly the simplest physical theory. There is a paucity of examples that can be formulated and studied in *complete detail*, though plenty of informal ideas and speculations have been aired. Classical mechanical systems offer interesting problems and insights into experimental computation.

Our result about arbitrary subsets $A \subset \mathbb{N}$ is new. Most attempts at undecidability embed recursively enumerable but non-recursive sets into models of physical systems, e.g., by simulating Turing machines and examining their halting problem. The fact that *any* subset of the natural numbers can be recognised by a simple mechanical system raises an alarm because the theory of the subsets of natural numbers is so vastly complicated it depends on the foundations of set theory for its exploration. Let us note that many sets of computational interest lie in the *arithmetic hierarchy*, which is a countable family of subsets of the natural numbers that already contains sets that are in a convincing sense *infinitely more undecidable* than the halting problem (see Rogers [46]).

Our bagatelles are systems that each require unbounded space, time and energy to decide $n \in A$ for all $n \in \mathbb{N}$. Consider energy. In each of our Newtonian bagatelles mass is bounded (indeed, it can be an arbitrary constant) and velocity is unbounded. In each of our Relativistic bagatelles velocity is bounded and mass is unbounded. One can ask if there are examples of kinematic systems that are bounded in space, time and energy?

In Newtonian mechanics we are allowed to shrink space and accelerate time. For example, the natural numbers $n = 0, 1, 2, \ldots$ that mark points in space or steps in time can be embedded into the interval $[0, 1]$ by $n \mapsto 1/2^n$. Shrinking space leads to mechanical systems that use arbitrarily small components. Of course, a mechanical system that exploits the infinite divisibility of space, with no *lower* bounds on units of space and time, violates any form of atomic theory. But such examples are sharp tools to investigate the theoretical foundations of computability and mechanics. In fact, it is possible to prove that for each set $A \subset \mathbb{N}$ there exists a valid Newtonian kinematic system S_A, which is embedded within a bounded 3-dimensional box, operates entirely within a fixed finite time interval using a fixed finite amount of energy, and can decide the membership of the subset A (Beggs and Tucker [5]).

However, an open problem is this:

Problem 6.1. *For all valid kinematic systems that possess both lower and upper bounds on space, time, mass, velocity and energy, are the sets and functions computable by experiment also computable by algorithms?*

We conjecture that the answer is "Yes". To prove this, one needs axiomatisations of the kind discussed in Section 6.2.

Our bagatelle examples show that the notion of mechanical system - i.e., what qualifies as a valid or legal system in theoretical mechanics - must be sharpened. To the standard parameters of mass, velocity, distance, time we need to add formal theory that constrains the structure and construction of the system and explains how experiments are performed.

Theoretical intuitions about making experiments turn out to be strikingly similar to intuitions about algorithms and computers, although the primitive actions are different and are implicit in the physical theory. Indeed, we conjecture that a theory of Gedanken experiments for mechanics, if formalised, could be capable of underpinning the theory of the computable as follows:

Problem 6.2. *Extend theoretical mechanics by a mathematical theory of construction and observation of mechanical systems, and show that the sets and functions computable by experiment are precisely those computable by algorithms.*

One goal of this direction of research, from physical theory to computablity, is, roughly speaking, *To derive forms of Church-Turing Thesis as physical laws.*

The bagatelle theorem is a theorem based on classical mathematical reasoning. It reveals the necessity of making explicit the nature of experiments in mechanical theorems. From the point of view of mathematical logic and computability theory, mechanical systems need specification languages that can describe formally their physical structure, construction and observation. However, from the point of view of philosophical foundations, we can reject classical mathematical reasoning, which allows the construction of such omnipotent mechanical systems, and use constructive mathematical reasoning. Here is another problem:

Problem 6.3. *Develop a constructive theoretical mechanics, based on constructive mathematical reasoning, in which experiments are part of the basis for mathematical existence.*

Finally, we should raise the special case of *efficient* computation by mechanical systems. New theory is needed to pose and answer a question such as:

Problem 6.4. *Are there sets that can be decided in polynomially bounded space and time by experimental computation with mechanical systems but cannot be decided by algorithms in polynomial space and time?*

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